STRUCTURAL THEOREMS FOR HOLOMORPHIC SELF-MAPS OF THE PUNCTURED PLANE

A thesis submitted by

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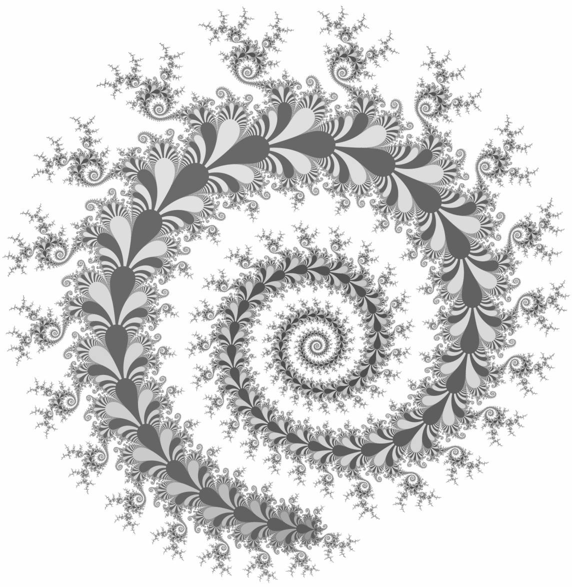
(Mathematics)



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DECLARATION

This thesis is the result of my own work, except where explicit reference is made to the work of others, and has not been submitted for another qualification to this or any other university.

Milton Keynes, September 16, 2016

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ABSTRACT

This thesis concerns the iteration of transcendental self-maps of the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, that is, functions $f : \mathbb{C}^* \to \mathbb{C}^*$ that are holomorphic on $\mathbb{C}^*$ and for which both zero and infinity are essential singularities. We focus on the *escaping set* of such functions, which consists of the points whose orbit accumulates to zero and/or infinity under iteration. The escaping set is closely related to the structure of the phase space due to its connection with the Julia set.

We introduce the concept of *essential itinerary* of an escaping point, which is a sequence that describes how its orbit accumulates to the essential singularities, and plays a very important role throughout the thesis. This allows us to partition the escaping set into uncountably many non-empty subsets of points that escape in non-equivalent ways, the boundary of each of which is the Julia set. We combine the iterates of the maximum and minimum modulus functions to define the *fast escaping set* for functions in this class and, for such functions, construct orbits with several types of *annular itinerary*, including fast escaping and arbitrarily slowly escaping points.

Next we proceed to study in detail the class $\mathcal{B}^*$ of *bounded-type* transcendental self-maps of $\mathbb{C}^*$, for which the escaping set is a subset of the Julia set, so such functions do not have escaping Fatou components. We show that, for finite compositions of transcendental self-maps of $\mathbb{C}^*$ of finite order (and hence in $\mathcal{B}^*$), every escaping point can be joined to one of the essential singularities by a curve of points that escape uniformly. Moreover, we prove that, for every essential itinerary, the corresponding escaping set contains a *Cantor bouquet* and, in particular, uncountably many such curves.

Finally, in the last part of the thesis we direct our attention to the functions that do have *escaping Fatou components*. We give the first explicit examples of transcendental self-maps of $\mathbb{C}^*$ with *Baker domains* and *escaping wandering domains* and use approximation theory to construct functions with escaping Fatou components that have any prescribed essential itinerary.

PUBLICATIONS

Much of the content of this thesis has previously appeared in the following papers:

(1) D. Martí-Pete, *The escaping set of transcendental self-maps of the punctured plane*, to appear in Ergodic Theory and Dynamical Systems, preprint arXiv:1412.1032.

(2) N. Fagella and D. Martí-Pete, *Dynamic rays of bounded-type transcendental self-maps of the punctured plane*, to appear in Discrete and Continuous Dynamical Systems - Series A, preprint arXiv:1603.03311.

(3) D. Martí-Pete, *Escaping Fatou components of transcendental self-maps of the punctured plane*, in preparation.

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While at Universitat de Barcelona, I coordinated the postgraduate seminar Seminari Informal de Matemàtiques de Barcelona (SIMBa). I also participated in the organisation of the networking event Jornada d'Investigadors Predoctorals Interdisciplinària (JIPI) that took place at UB on February 7, 2013. I am grateful to all the people that got involved in these activities.

I would not be here if Jordi Taixés had not introduced me to the beauty of this research area in the first year of my undergraduate degree at Universitat de Barcelona, asking us to iterate Newton's method to find the centres of hyperbolic components of the Mandelbrot set, thank you!

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Finally, I thank my parents for encouraging me to pursue my dreams despite, sometimes, this requires being far from home. This thesis is dedicated to them.

CONTENTS

LIST OF FIGURES

1

INTRODUCTION

The material in this thesis belongs to the research area known as Complex Dynamics, which lies in the intersection between Dynamical Systems and Complex Analysis. The iteration of rational and transcendental entire functions has been widely studied since the early 20th century. However, the iteration of transcendental self-maps of the punctured plane, that is, holomorphic self-maps of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ that have *two* essential singularities, at zero and infinity, has received much less attention. In this thesis we study the escaping set of such functions, which consists of the points that accumulate at zero and/or infinity under iteration.

1.1 AN INTRODUCTION TO COMPLEX DYNAMICS

Complex Dynamics concerns the iteration of a holomorphic function on a Riemann surface S, usually a subset of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. If $f : S \subseteq \hat{\mathbb{C}} \to S$ is holomorphic and $\hat{\mathbb{C}} \setminus S$ consists of essential singularities, then conjugating by a Möbius transformation, we can reduce to the following three cases:

- $S = \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and f is a rational map;
- $S = \mathbb{C}$ and f is a transcendental entire function;
- $S = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and f has essential singularities at *both* zero and infinity.

By Picard's theorem, there are no holomorphic self-maps of $S \subseteq \hat{\mathbb{C}}$ where $\hat{\mathbb{C}} \setminus S$ consists of three or more essential singularities. Observe that, for the same reason, holomorphic self-maps of $\mathbb{C}^*$ have no omitted values in $\mathbb{C}^*$. We may also consider the iteration of *transcendental meromorphic functions* on $\mathbb{C}$, for which infinity is an essential singularity and the poles form a discrete set. The texts [Bea91; CG93; Mil06; Ste93] are basic references on the iteration of holomorphic functions;

see [Ber93] for a survey on the iteration of transcendental entire and meromorphic functions.

The dynamical partition

Given a holomorphic function $f : S \to S$, where S is $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$, and a point $z_0 \in S$, we consider the sequence given by its iterates

$$f^n(z_0) := (f \circ \overset{n}{\cdots} \circ f)(z_0) \quad \text{for } n \in \mathbb{N},$$

and, for $n = 0$, we define $f^0(z_0) := z_0$. Throughout the thesis, we use the notation $\mathbb{N}_0$ for the set $\mathbb{N} \cup \{0\}$. We define the *(forward) orbit* of a point $z_0 \in S$ to be the set

$$\mathcal{O}^+(z_0, f) := \{z \in S \ : \ z = f^n(z_0) \text{ for some } n \in \mathbb{N}_0\}$$

and the *backward orbit* of $z_0 \in \hat{\mathbb{C}}$ to be the set

$$\mathcal{O}^-(z_0, f) := \{z \in S \ : \ z_0 = f^n(z) \text{ for some } n \in \mathbb{N}\}.$$

We also define the *grand orbit* of $z_0 \in S$ to be the set

$$\mathcal{O}(z_0, f) := \{z \in S \ : \ f^m(z) = f^n(z_0) \text{ for some } m, n \in \mathbb{N}\}.$$

We say that a set $X \subseteq S$ is *(forward) invariant* under f if $f(X) \subseteq X$. If, moreover, $f^{-1}(X) \subseteq X$, then we say that X is *completely invariant* under f. Grand orbits are the smallest completely invariant sets that we can partition the phase space, or set of initial conditions, S into. However, we will be interested in another partition that arises from the dynamics.

We define the *Fatou set* of f, or *stable set*, as

$$F(f) := \left\{z \in S \ : \ \{f^n\}_{n \in \mathbb{N}} \text{ is a normal family in a neighbourhood of } z\right\}$$

and we define the *Julia set* of f, or *chaotic set*, as its complement, $J(f) := S \setminus F(f)$. Alternatively, we can define $F(f)$ as the set of points where the family of iterates of f are equicontinuous. Thus, the Fatou set is open and the Julia set is closed.

The sets $F(f)$ and $J(f)$ are named after Pierre Fatou (1878 – 1929) and Gaston Julia (1893 – 1978). Motivated by the 1918 Grand Prix of the Académie des Sciences in Paris which was to be awarded for a study of iteration from a global point of view, both of these French mathematicians produced long memoirs on the use of Montel's normal families in the iteration of holomorphic functions [Fat19; Jul18]. Previously, Schröder and Cayley had studied the dynamics of Newton's method applied to a cubic polynomial but failed to give a global description because this involves what we now know to be a fractal Julia set. Although Fatou ended up not submitting his work and Julia was awarded the prize, their works laid the foundations of what we now call Complex Dynamics. You can read about the beginnings of this research area in [Ale94].

Both Fatou and Julia studied the iteration of rational functions in their memoirs and later on Fatou [Fat26] also studied the iteration of transcendental entire functions. However, it was not until 1953 that Rådström [Råd53] considered the iteration of holomorphic self-maps of $\mathbb{C}^*$. In this section we outline the basic properties of the iteration of rational functions, transcendental entire functions and transcendental self-maps of $\mathbb{C}^*$, and in Section 1.3 we describe the aspects of research which are specific to holomorphic self-maps of $\mathbb{C}^*$.

The Fatou and Julia sets are both completely invariant and satisfy $F(f^n) = F(f)$ and $J(f^n) = J(f)$ for $n \in \mathbb{N}$ (see [Ber93, Section 1.2] for the proofs of the elementary properties of the Fatou and Julia sets). The Julia set has the dichotomy that either $J(f) = S$ or $J(f)$ has empty interior, and there are examples of functions for which $J(f) = S$ in the three classes described above. We say that $z_0 \in S$ is a *(finite) exceptional value* if $\mathcal{O}^-(z_0, f)$ is finite. Rational functions have at most two such values while transcendental entire functions have at most one and transcendental self-maps of $\mathbb{C}^*$ have none. If $z_0 \in J(f)$ is not an exceptional value, then we have $J(f) = \overline{\mathcal{O}^-(z_0, f)}$. It follows from Montel's theorem and the existence of repelling periodic points in $J(f)$ (that we will discuss in the next section) that if $z_0 \in J(f)$ and U is an open neighbourhood of z_0, then for any compact set $K \subseteq S$ which contains no exceptional values, there exists $N = N(K) \in \mathbb{N}$ such that $f^n(U) \supseteq K$ for all $n \geqslant N$. We refer to this property as the *blowing-up property* of the Julia set.

Periodic orbits

We say that a point $z_0 \in S$ is a *fixed point* of f if $f(z_0) = z_0$ and we say that z_0 is a *periodic point* of f (of period $p \in \mathbb{N}$) if $f^p(z_0) = z_0$ and $f^k(z_0) \neq z_0$ for $0 < k < p$. Observe that periodic points of period p can be regarded as fixed points of the map f^p. If there exists $n \in \mathbb{N}$ such that $f^n(z_0)$ is a periodic point but z_0 is not periodic, then we say that z_0 is a *preperiodic point* of f.

Given a periodic cycle $\{z_0, z_1, \ldots, z_{p-1}\} \subseteq \mathbb{C}$ of f of period $p \in \mathbb{N}$; that is, $f(z_k) = z_{k+1}$ for $0 \leqslant k < p-1$ and $f(z_{p-1}) = z_0$, we define the *multiplier* of this orbit as

$$\lambda := (f^p)'(z_k) = \prod_{j=0}^{p-1} f'(z_j)$$

where $0 \leqslant k < p$. Observe that the second equality follows easily from the chain rule. The multiplier of a periodic cycle is invariant under conformal conjugation, so if a periodic cycle contains infinity, we can conjugate f by a Möbius transformation and reduce to the case above. We also refer to λ as the *multiplier* of each point z_k, $0 \leqslant k < p$, in the orbit. Note that for fixed points the multiplier is just the derivative of the function at that point. Then, according to the multiplier, we classify periodic orbits into the following types:

- we say that $\{z_0, z_1, \ldots, z_{p-1}\}$ is a *(super)attracting* periodic orbit if $|\lambda| < 1$ ($\lambda = 0$), and then there exists a neighbourhood U_k of each point z_k, $0 \leqslant k < p$, such that for all $z \in U_k$, $f^{np}(z) \to z_k$ as $n \to \infty$;

- we say that $\{z_0, z_1, \ldots, z_{p-1}\}$ is a *repelling* periodic orbit if $|\lambda| > 1$, and then there exists a neighbourhood U_k of each point z_k, $0 \leqslant k < p$, such that for all $z \in U_k \setminus \{z_k\}$, there exists a value $n = n(z) \in \mathbb{N}$ such that $f^{np}(z) \notin U_k$;

- we say that $\{z_0, z_1, \ldots, z_{p-1}\}$ is an *indifferent* (or *neutral*) periodic orbit if $|\lambda| = 1$, and then we can further classify the orbit into:
  - a *rationally indifferent* (or *parabolic*) periodic orbit if $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{Q}$;
  - an *irrationally indifferent* periodic orbit if $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Observe that attracting periodic orbits are in the Fatou set, while repelling and rationally indifferent periodic points are in the Julia set. Irrationally indifferent periodic orbits can lie in either $F(f)$ or $J(f)$ depending on the arithmetic properties of the number $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Both Fatou and Julia showed that, for rational functions, $J(f)$ is the closure of the set of repelling periodic points of f. For transcendental entire functions, this fact is due to Baker [Bak68] and, for transcendental self-maps of $\mathbb{C}^*$, this was proved by Bhattacharyya [Bha69]. Results on the existence of periodic points can be used to show that $J(f)$ is an infinite set and it then follows that $J(f)$ is a perfect set.

We mentioned before that the set $J(f)$ is chaotic. To justify this we use the definition given by Devaney [Dev86]: a dynamical system $f : X \to X$ is said to be *chaotic* if it has the following three properties:

- f has *sensitive dependence on initial conditions*; that is, there exists $\delta > 0$ such that for any $p \in X$ and any open neighbourhood U of p, there exist $q \in U$ and $n \geqslant 0$ such that $|f^n(p) - f^n(q)| > \delta$;
- f is *topologically transitive*; that is, for every pair of non-empty open sets $U, V \subseteq X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$;
- periodic orbits are dense in X.

Observe that our previous discussion implies that the restriction of f to $J(f)$ satisfies these three properties.

Classification of the Fatou components

We now turn our attention to the Fatou set. We refer to the connected components of the Fatou set as *Fatou components*. Let U be a Fatou component and denote by U_n, $n \in \mathbb{N}$, the Fatou component that contains $f^n(U)$. For rational functions, we have $f(U) = U_1$. Herring [Her98] showed that, for transcendental entire functions and transcendental self-maps of $\mathbb{C}^*$, $U_1 \setminus f(U)$ consists of at most one point, which need not be an omitted value. Moreover, he showed that, for transcendental self-maps of $\mathbb{C}^*$, if U_1 is doubly connected, then $f(U) = U_1$.

We say that a Fatou component U is *periodic* if $U_p = U$ for some $p \in \mathbb{N}$ and we say that U is *preperiodic* if U is not periodic but U_n is periodic for some $n \in \mathbb{N}$. Suppose that U is an invariant Fatou component of f (otherwise consider an iterate f^p). Then U can be classified into the following types:

- U is an *(immediate) basin of attraction*: there exists an attracting fixed point $z_0 \in U$ and for all $z \in U$, $f^n(z) \to z_0$ as $n \to \infty$;
- U is a *parabolic basin of attraction* (or a *Leau domain*): there exists a parabolic fixed point $z_0 \in \partial U$ with multiplier $\lambda = 1$ and for all $z \in U$, $f^n(z) \to z_0$ as $n \to \infty$;
- U is a *Siegel disc*: there exists an irrationally indifferent fixed point $z_0 \in U$ with multiplier $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exists a biholomorphic function $\phi : U \to \mathbb{D}$ such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\theta} z$ for all $z \in \mathbb{D}$;
- U is a *Herman ring*: there exists a biholomorphic function $\phi : U \to A$ for some annulus $A := \{z \in \mathbb{C} \ : \ 1 < |z| < r\}$ such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\theta} z$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and all $z \in A$;
- U is a *Baker domain*: there exists an essential singularity $\alpha \in \partial U$ and for all $z \in U$, $f^n(z) \to \alpha$ as $n \to \infty$.

It follows easily from the maximum principle that Herman rings must contain a pole or an essential singularity inside their bounded complementary component, and hence entire functions do not have Herman rings. Since Baker domains require the presence of an essential singularity, rational functions do not have Baker domains.

We say that a Fatou component U is a *wandering domain* if U is neither periodic nor preperiodic or, in our previous notation, if $U_m = U_n$ implies $m = n$. Fatou conjectured that rational functions do not have wandering domains but was not able to discard this possibility. After the papers of Fatou [Fat19] and Cremer [Cre32] studying periodic Fatou components, we have to wait for over 50 years until Sullivan's celebrated 'no wandering domains' theorem [Sul85], in which he proved Fatou's conjecture. However, transcendental entire functions and transcendental self-maps of $\mathbb{C}^*$ do have wandering domains (see [Bak63; Bak76] and [Kot90] respectively).

Following Sullivan's result, Eremenko and Lyubich [EL84] and Goldberg and Keen [GK86] proved, independently, that if f is a transcendental entire function of finite type (see definition later), then $F(f)$ has no wandering domains. The analogous statement for transcendental self-maps of $\mathbb{C}^*$ was proved, also independently, by Keen [Kee88] and Kotus [Kot87] (see also [Mak87] and [Fan91]).

Fatou [Fat19] proved that a rational function can have at most *two* completely invariant Fatou components. For transcendental entire functions, Baker [Bak70] showed that there can be at most *one* completely invariant Fatou component and Bhattacharyya [Bha83] adapted Baker's arguments to $\mathbb{C}^*$. Later on, Hinkkanen [Hin94] gave an alternative proof of this result for $\mathbb{C}^*$.

Singularities of the inverse function

There is a very strong connection between the Fatou components of f and the singularities of the inverse function f^{-1}. We denote by $\operatorname{sing}(f^{-1})$ the set of singularities of f^{-1} which consists of the following two kinds of points:

- $v \in S$ is a *critical value* of f if there exists a critical point $c \in S$ (that is, $f'(c) = 0$) such that $v = f(c)$;
- $a \in S$ is a *(finite) asymptotic value* of f if there exists a curve $\gamma : [0, +\infty) \to S$ (an *asymptotic path* over a) such that $\gamma(t) \to \alpha$ as $t \to +\infty$ where α is an essential singularity of f and $f(\gamma(t)) \to a$ as $t \to +\infty$.

Of course, rational functions do not have asymptotic values and have a finite number of critical values. In the entire case, we define the *Speiser class* of *finite-type* transcendental entire functions by

$$\mathcal{S} := \{f \text{ transcendental entire function } : \#\operatorname{sing}(f^{-1}) < +\infty\}.$$

Note that, for transcendental self-maps of $\mathbb{C}^*$, asymptotic paths can either tend to zero or to infinity. The following results relate Fatou components and singular values of a function f:

- if $\{U_0, U_1, \dots, U_{p-1}\}$ is a cycle of immediate basins of attraction or a cycle of parabolic basins of attraction, then there exists $0 \leqslant k < p$ such that $U_k \cap \operatorname{sing}(f^{-1}) \neq \emptyset$;
- if $\{U_0, U_1, \dots, U_{p-1}\}$ is a cycle of Siegel discs or a cycle of Herman rings, then $\partial U_k \subseteq \overline{\mathcal{O}^+(\operatorname{sing}(f^{-1}), f)}$ for all $0 \leqslant k < p$.

The connection between Baker domains and wandering domains and the singularities of the inverse is more subtle (see, for instance, [BW91; EL92; BD99; MR13]).

1.2 THE ESCAPING SET

For an entire function f, we define the *escaping set* of f as the set of points that tend to infinity under iteration; that is,

$$I(f) := \{z \in \mathbb{C} \;:\; f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Despite being completely different from the topological point of view, the investigation of the properties of the escaping set has provided important insight into the Julia set of both polynomials and transcendental entire functions.

The escaping set of polynomials

For polynomials, the escaping set consists of the basin of attraction of infinity, which is an unbounded connected open set in $F(f)$, and its boundary is $J(f)$. In this setting, the complement of $I(f)$ is known as the *filled Julia set*, $K(f)$. In the Orsay notes [DH84], Douady and Hubbard carried out an extensive study of the dynamics of quadratic polynomials. One of the main tools used there are external rays: given a connected filled Julia set $K(f)$, we can define the *Böttcher map*

$$\varphi_{K(f)} : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \overline{\mathbb{D}},$$

which is conformal, and then, for every $\theta \in [0, 1)$, define the *external ray* of argument θ as

$$R_\theta(r) := \varphi^{-1}_{K(f)}(re^{2\pi i\theta}), \quad \text{for } r > 1.$$

The external ray R_θ is said to *land* if there exists $z_\theta \in K(f)$ such that $R_\theta(r) \to z_\theta$ as $r \to 1$. By Carathéodory's theorem, if $K(f)$ is locally connected, then all external rays land. Thus the fact that external rays are organised by the dynamics $f(R_\theta) = R_{d\theta \pmod 1}$, where $d = \deg P$, leads to a combinatorial description of the Julia set.

The escaping set of transcendental entire functions

For transcendental entire functions, the escaping set also plays a very important role although the nature of the set is much more complicated. It was first studied by Eremenko [Ere89] who used Wiman-Valiron theory to show that, for a transcendental entire function f,

(I1) $I(f) \cap J(f) \neq \emptyset$;

(I2) $J(f) = \partial I(f)$;

(I3) the components of $\overline{I(f)}$ are unbounded.

We often refer to properties (I1), (I2) and (I3) as *Eremenko's properties*. Note that property (I1) contrasts strongly with the situation for polynomials, whereas property (I2) is common for both polynomials and transcendental entire functions. In the same paper, Eremenko conjectured that, for transcendental entire functions, property (I3) can be strengthened to say that all the components of the escaping set are also unbounded and this remains an open question.

A stronger version of Eremenko's conjecture states that, for a transcendental entire function, every escaping point a can be joined to infinity by a curve of points that escape uniformly. Such curves are called *ray tails* and their maximal extensions are called *dynamic rays* (see Definition 3.54) in analogy to the polynomial case.

Devaney and Krych [DK84] showed that for certain maps in the *exponential family*

$$E_\lambda(z) := \lambda e^z, \quad \lambda \in \mathbb{C}^*,$$

namely if $\lambda \in (0, 1/e)$, the Julia set of E_λ consists of dynamic rays that they called *hairs* (see Figure 1). Devaney and Tangerman [DT86] proved that the same holds for certain functions of *finite type*, that is, functions with finitely many singular values, satisfying additional technical conditions, such as the sine family $S_\lambda(z) = \lambda \sin(z), \lambda \in (0, 1)$. They coined the term *Cantor bouquet* to describe the Julia set of these functions. They first defined a Cantor N-bouquet, where $N \in \mathbb{N}$, to be a subset of $J(f)$ homeomorphic to the product of a Cantor set Σ_N and the half-line $[0, +\infty)$ satisfying that $\Sigma_{N_1} \subseteq \Sigma_{N_2}$ if $N_1 \leqslant N_2$, and then they defined a Cantor bouquet to be an increasing union of Cantor N-bouquets as $N \to \infty$.

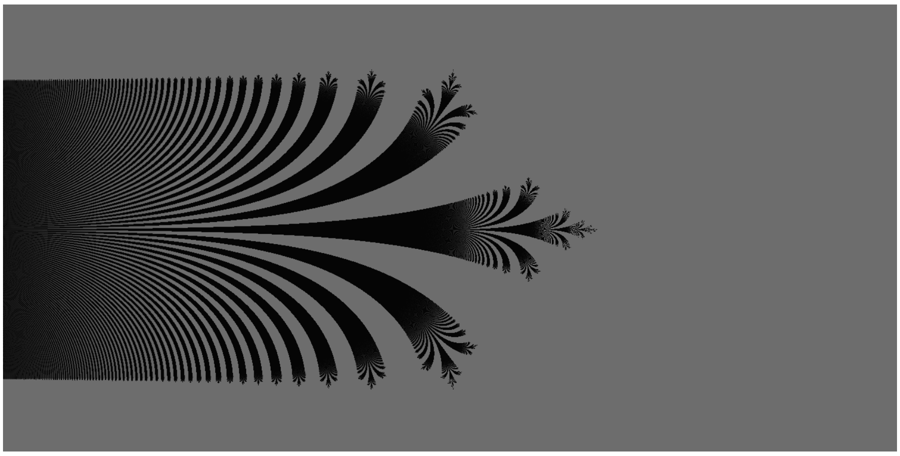


Figure 1: Cantor bouquet in the phase space of the function $f(z) = z+1+e^{-z}$ originally studied by Fatou.

Aarts and Oversteegen [AO93] introduced a slightly different definition of a Cantor bouquet in terms of a topological object called a *straight brush* (see Definition 3.68) that allows the comparison of Julia sets of different functions, for example, showing that they are homeomorphic and are equivalently embedded in the plane.

The Eremenko-Lyubich class $\mathcal{B}$

Eremenko and Lyubich [EL92] studied the dynamics of functions in the class

$$\mathcal{B} := \{f \text{ trancendental entire function } : \operatorname{sing}(f^{-1}) \text{ is bounded}\},$$

which includes all the functions in the class $\mathcal{S}$ and many more; we say that these functions have *bounded type*. They showed that, besides Eremenko's properties (I1), (I2) and (I3), functions in the class $\mathcal{B}$ additionally satisfy

(I4) $I(f) \subseteq J(f)$;

or, in other words, they have no Baker domains and no escaping wandering domains. In terms of Iversen's classification of singularities [Ive14] (see also [BE95]), functions in the class $\mathcal{B}$ have a *direct logarithmic singularity* over infinity: if $R > 0$ is sufficiently large that $\operatorname{sing}(f^{-1}) \subseteq D(0, R)$ and $W := \mathbb{C} \setminus \overline{D(0, R)}$, then each connected component V of the set $\mathcal{V} := f^{-1}(W)$ is an unbounded Jordan domain called a *(logarithmic) tract* of f, and the restriction $f_{|V} : V \to W$ is a universal covering map. To study the properties of the escaping set of functions in the class $\mathcal{B}$, Eremenko and Lyubich introduced a logarithmic change of variables in a neighbourhood of infinity, also known as *logarithmic coordinates*. Take $R > 0$ so that $|f(0)| < R$; if we define $\mathcal{T} := \exp^{-1} \mathcal{V}$ and $H := \exp^{-1} W$, then there exists a holomorphic function $F : \mathcal{T} \to H$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ \mathcal{V} \arrow[r, "f"'] & W \end{tikzcd}$$

Such a function F is called a *logarithmic transform* of f and the restriction $F_{|T} : T \to H$ is a conformal isomorphism for every component T of $\mathcal{T}$ (also called a *tract* of F). Using this, Eremenko and Lyubich showed that functions in the class $\mathcal{B}$ have a strong *expansivity property* (see Lemma 3.21), which enabled them to prove (I4).

To study the dynamics of F, it is useful to consider the set of points whose orbit under F is contained in $\mathcal{T}$ and use *symbolic dynamics* to describe their orbits: to every point z, we can associate a sequence of tracts (T_n), the *external address* of z, so that $F^n(z) \in T_n$ for $n \in \mathbb{N}_0$. Then the iteration of F in this set corresponds to the iteration of the Bernoulli shift map σ given by $(T_n) \mapsto (T_{n+1})$ on the set of infinite sequences $\mathcal{T}^{\mathbb{N}_0}$.

Functions in the class $\mathcal{B}$ satisfy many other useful properties. For instance, they are bounded on a path to infinity, and hence all their Fatou components are simply connected, by Baker [Bak84], and it follows from a theorem of Heins [Hei48] that they have lower order (see definition in Section 3.1) at least a half.

In 2011, Rottenfußer, Rückert, Rempe and Schleicher [RRRS11] proved that the stronger version of Eremenko's conjecture holds for transcendental entire functions of bounded type and finite order or, more generally, for finite compositions of such functions. Roughly speaking, we say that an entire function has *finite order* if the maximum modulus of f in the disc $D(0, r)$ does not grow faster than $\exp(r^k)$ as $r \to +\infty$ for some $k \in \mathbb{N}$. In the second part of [RRRS11], the authors show that there is a function in the class $\mathcal{B}$ for which every path-connected component of $J(f)$ (and thus $I(f)$) is bounded, and hence the stronger version of Eremenko's conjecture fails in $\mathcal{B}$. The positive result from [RRRS11] was proved independently by Barański [Bar07] for functions of *disjoint type*, that is, transcendental entire functions for which the Fatou set consists of a single completely invariant component that is a basin of attraction. Later on, Barański, Jarque and Rempe [BJR12] proved that, actually, the Julia set of the functions considered in [RRRS11] contains a Cantor bouquet.

The fast escaping set

Key progress on Eremenko's conjecture was obtained by studying the *fast escaping set* defined by

$$A(f) := \{z \in \mathbb{C} \ : \ \exists \ell \in \mathbb{N}_0, \ |f^{n+\ell}(z)| \geqslant M^n(R, f) \text{ for all } n \in \mathbb{N}_0\},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$ and $R > 0$ is chosen to be sufficiently large so that $M^n(R, f) \to +\infty$ as $n \to \infty$.

The set $A(f)$, which consists of the points that escape about as fast as possible, was introduced by Bergweiler and Hinkkanen [BH99] and shares some properties with $I(f)$, for example, $J(f) \cap A(f) \neq \emptyset$ and $J(f) = \partial A(f)$ (see [BH99] and [RS05b]). But it also has some much nicer properties. Rippon and Stallard showed that all the components of $A(f)$ are unbounded, and hence $I(f)$ has at least one unbounded component. For the class of functions from [RRRS11], Rippon, Rempe and Stallard [RRS10] proved that, under an additional condition, the dynamic rays in $J(f)$ are in $A(f)$ apart from possibly their finite endpoint. The paper [RS12] gives a compilation of results about the fast escaping set.

For every transcendental entire function, the Julia set also contains points that escape arbitrarily slowly; that is, for any sequence (r_n) of positive real numbers such that $r_n \to +\infty$ as $n \to \infty$, there is $z \in J(f)$ such that $|f^n(z)| \leqslant r_n$ for all $n \in \mathbb{N}$ sufficiently large. This follows, for example, from the construction in [RS15]. In that paper, Rippon and Stallard study a partition of the plane into annuli that are defined using the iterates of the maximum modulus function $M(r, f)$ starting with a sufficiently large $r > 0$. To each point $z \in I(f)$, they associate a sequence of natural numbers (s_n), the *annular itinerary*, that describes to which annulus the nth iterate of z belongs. Rather surprisingly, they then show that, for every transcendental entire function, almost every possible sequence is the annular itinerary of a point in $I(f)$. For the class of functions studied in [RRS10], $J(f)$ consists of dynamic rays and the slow escaping points are a subset of the endpoints of these rays.

The escaping set of other classes of functions

To conclude this section, we define the escaping set for other classes of functions that are not entire. If f is a transcendental meromorphic function, then we define the escaping set of f by

$$I(f) := \{z \in \mathbb{C} \ : \ f^n(z) \text{ is defined for } n \in \mathbb{N} \text{ and } f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Note that if f is a holomorphic self-map of $\mathbb{C}^*$ with an essential singularity at infinity and a single pole at the origin that is omitted, then we can use the same definition of $I(f)$ as for entire functions. Domínguez [Dom98] proved that the analogues of Eremenko's properties (I1) and (I2) also hold in this setting; that is, $I(f) \cap J(f) \neq \emptyset$ and $J(f) = \partial I(f)$. However, in this case the components of $\overline{I(f)}$ need not be unbounded but, if they are bounded, then they need to have the pole at zero in their closure.

For transcendental self-maps of $\mathbb{C}^*$, the escaping set is given by

$$I(f) := \{z \in \mathbb{C}^* \ : \ \omega(z, f) \subseteq \{0, \infty\}\}$$

where $\omega(z, f)$ is the classical omega-limit set

$$\omega(z, f) := \bigcap_{n \in \mathbb{N}} \overline{\{f^k(z) \ : \ k \geqslant n\}},$$

and the closure is taken in $\hat{\mathbb{C}}$. Very little has been proved about this set, which forms the focus of our thesis.

1.3 HOLOMORPHIC SELF-MAPS OF THE PUNCTURED PLANE

In this section we describe the properties of the iteration of transcendental self-maps of $\mathbb{C}^*$ and the larger class of holomorphic self-maps of $\mathbb{C}^*$. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a holomorphic function and suppose that

f is not a rational function. Then we can assume without loss of generality that infinity is an essential singularity and f can be classified into one of the following three classes according to the nature of the point zero:

1. zero is a regular point and f is a transcendental entire function, and hence $f(z) = z^n \exp(g(z))$, where $n \geqslant 0$ and g is a non-constant entire function;

2. zero is a pole and f is a transcendental meromorphic function, and hence $f(z) = z^n \exp(g(z))$, where $n < 0$ and g is a non-constant entire function;

3. zero is an essential singularity, and hence

$$f(z) = z^n \exp(g(z) + h(1/z)), \tag{1.1}$$

where $n \in \mathbb{Z}$ and g, h are non-constant entire functions.

The expressions for the function f above follow from the fact that the function $\log(f(z)/z^n)$ is holomorphic in $\mathbb{C}^*$ (see [Råd53, p. 88] and [Bha69, Section 1.2] for the details).

The number $n \in \mathbb{Z}$ in the cases 1-3 above is called the *index of* f, written $\operatorname{ind}(f) = n$, and equals the index (or winding number) of $f(\gamma)$ with respect to the origin, where γ is any positively oriented simple closed curve around the origin.

If f is a holomorphic self-map of $\mathbb{C}^*$, then there exists a transcendental entire function $\tilde{f}$ that is semiconjugated to f by the exponential function; that is,

$$\exp \circ \tilde{f} = f \circ \exp .$$

Such a function $\tilde{f}$ is called a *lift of* f and is unique up to the addition of multiples of $2\pi i$; that is, $\tilde{f}(z) + 2k\pi i$ is also a lift of f for any $k \in \mathbb{Z}$. For example, if $f(z) = z^n \exp(z + 1/z)$ with $n \in \mathbb{Z}$, then the entire function $\tilde{f}(z) = nz + e^z + e^{-z} = nz + 2\cosh(z)$ is a lift of f. If $\tilde{f}$ is a lift of f, then

$$\tilde{f}(z + 2k\pi i) = \tilde{f}(z) + \operatorname{ind}(f) \cdot 2k\pi i \tag{1.2}$$

for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$. Bergweiler [Ber95] showed that if f is a holomorphic self-map of $\mathbb{C}^*$ and $\tilde{f}$ is a lift of f, then $J(\tilde{f}) = \exp^{-1} J(f)$.

Examples of holomorphic self-maps of $\mathbb{C}^*$

Transcendental self-maps of $\mathbb{C}^*$ arise in a natural way in many instances, for example, when you complexify circle maps, like the so-called *Arnol'd standard family*, by performing analytic continuation:

$$f_{\alpha\beta}(\theta) = \theta + \alpha + \beta \sin\theta \pmod{2\pi}, \quad 0 \leqslant \alpha \leqslant 2\pi,\ \beta \geqslant 0,$$

for $\theta \in \mathbb{R}$, has as complexification (see, for example, [Fag99]):

$$\hat{f}_{\alpha\beta}(z) = ze^{i\alpha}e^{\beta(z-1/z)/2}, \quad 0 \leqslant \alpha \leqslant 2\pi,\ \beta \geqslant 0,$$

for $z \in \mathbb{C}^*$; that is, $\hat{f}_{\alpha\beta}(e^{i\theta}) = e^{if_{\alpha\beta}(\theta)}$ for all $\theta \in \mathbb{R}$ (see Figure 4). We refer to the functions $\hat{f}_{\alpha\beta}$ as the *complex standard family*.

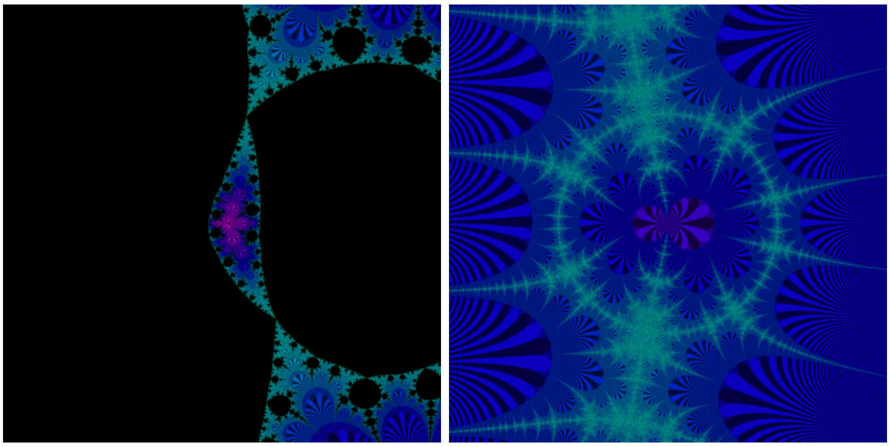


Figure 2: Phase space of the function $\hat{f}_{\alpha\beta}$ from the complex standard family, with $\alpha = 3.1,\ \beta = 0.8$ (left) and $\alpha = 3.1,\ \beta = 5$ (right).

Fagella [Fag99] showed that for maps in the complex standard family there exists an invariant set of dynamic rays that are organised by some symbolic dynamics and consists of points that, except possibly the finite endpoints, escape exponentially fast. For parameters with rational rotation numbers, that is, in the so-called Arnol'd tongues, she characterised some bifurcations in terms of sets of hairs attaching to the unit circle. Finally, she also showed that, for some irrational rotation numbers, the Fatou set contains a Herman ring.

The maps in the complex standard family belong to the class of holomorphic self-maps of $\mathbb{C}^*$ that are of the form

$$f(z) = z^n \exp(P(z) + Q(1/z)) \tag{1.3}$$

where $n \in \mathbb{Z}$ and P, Q are polynomials. Such maps were considered for the first time by Keen [Kee88; Kee89] who showed that they have a finite number of singular values. Later on we will see that transcendental self-maps of $\mathbb{C}^*$ of finite order are precisely of this form. Finite type holomorphic self-maps of $\mathbb{C}^*$ were also studied in [Kee88; Kot87; Mak91], where the authors investigated the space of functions that are topologically conjugated to a given transcendental self-map of $\mathbb{C}^*$.

Properties of the Fatou set

One of the main differences between the iteration of transcendental entire functions and transcendental self-maps of $\mathbb{C}^*$ lies in the topology of their Fatou components. Baker [Bak87, Theorem 1] (see also [Mak87] and [Fan91]) showed that if f is a holomorphic self-map of $\mathbb{C}^*$ that is not a rational function, then the components of $F(f)$ are either simply connected or doubly connected, and there is at most one doubly connected component, which must separate zero from infinity. This contrasts with the fact that, for transcendental entire functions, Baker [Bak84] proved that every multiply connected Fatou component is a wandering domain whose iterates escape to infinity.

Another difference between iteration in $\mathbb{C}$ and in $\mathbb{C}^*$ is that, in the entire case, multiply connected Fatou components are bounded while, for transcendental self-maps of $\mathbb{C}^*$, doubly connected Fatou components may be bounded or unbounded. We say that a set X is *unbounded in* $\mathbb{C}^*$ if $\hat{X} \cap \{0, \infty\} \neq \emptyset$, where $\hat{X}$ is the closure of X in $\hat{\mathbb{C}}$. Baker showed that the functions

$$f(z) = \exp(\alpha z - \alpha/z), \quad 0 < \alpha < 1/2, \tag{1.4}$$

have a doubly connected Fatou component with both zero and infinity in its boundary (see [Bak87, Theorem 2]).

Later on, Baker and Domínguez [BD98] classified the doubly connected Fatou components of holomorphic self-maps of $\mathbb{C}^*$ that are bounded in $\mathbb{C}^*$. Let U be such a Fatou component of a function f. Then there are three possibilities:

- U is an invariant Herman ring;

- U is a preperiodic Fatou component;
- U is a wandering domain.

The first case can only occur if $|\text{ind}(f)| = 1$, while the second and third cases can only occur if $\text{ind}(f) = 0$. Therefore, if $\text{ind}(f) \notin \{0, \pm 1\}$, the bounded Fatou components of f are all simply connected (see [Mak91]).

Since it was not clear whether the doubly connected Fatou component was bounded or not for the previous examples in the literature, Baker and Domínguez provided examples of functions of each of the three kinds and showed that their doubly connected Fatou components are bounded [BD98, Theorems 5, 6 and 7]. For instance, they constructed a function with an attracting basin that has a doubly connected preimage that is bounded in $\mathbb{C}^*$.

Baker and Domínguez [BD98] also discussed the case of unbounded doubly connected Fatou components. They observed that it is possible to have periodic Fatou components that are doubly unbounded, and gave as an example (1.4) whose Fatou set is connected and hence unbounded in $\mathbb{C}^*$, and completely invariant.

Baker [Bak87] used approximation theory to construct the first example of a holomorphic self-map of $\mathbb{C}^*$ (which was entire) with a wandering domain that escapes to infinity. The first examples of transcendental self-maps of $\mathbb{C}^*$ with a wandering domain are due to Kotus [Kot90], where the wandering domain accumulates to zero or infinity, or alternates between both of them. In the same paper, Kotus also constructed an example with an infinite limit set by adapting the techniques from [EL92]. Mukhamedshin [Muk91] used quasiconformal surgery to glue together two transcendental entire functions, each with a Siegel disc and a wandering domain, and thus create a transcendental self-map of $\mathbb{C}^*$ with a Herman ring and two wandering domains, one escaping to zero and the other one to infinity. And finally, the most recent examples are due to Baker and Domínguez [BD98] who constructed examples of transcendental self-map of $\mathbb{C}^*$ with doubly connected wandering domains that escape to infinity and can be chosen to be bounded or unbounded in $\mathbb{C}^*$.

The only previous examples of Baker domains of transcendental self-maps of $\mathbb{C}^*$ that the author is aware of are due to Kotus [Kot90]. She used approximation theory to construct two functions with invariant hyperbolic Baker domains escaping to zero and to infinity respectively.

Properties of the Julia set

Regarding the properties of the Julia set, Baker and Domínguez [BD98] proved that, if f is a transcendental self-map of $\mathbb{C}^*$, then all the components of $J(f)$ are unbounded in $\mathbb{C}^*$. Note that, in particular, this implies that $J(f)$ does not have singleton components. This contrasts with the following result from Domínguez [Dom97] (see also [Ber00]), who proved that if a transcendental entire function has a multiply connected Fatou component, then buried singleton components are dense in $J(f)$; a component of $J(f)$ is called *buried* if it does not meet the boundary of any Fatou component of f. However, Kisaka [Kis98] showed that this phenomenon is specific to transcendental entire functions with multiply connected Fatou components. He proved that, for a transcendental entire function f, all the components of $J(f)$ are unbounded if and only if f has no multiply connected Fatou component.

For transcendental entire or meromorphic functions, the Julia set has either one or uncountably many components (see, for example, Baker and Domínguez [BD00]). Baker and Domínguez [BD98] proved that, if f is a holomorphic self-map of $\mathbb{C}^*$ that is not a rational function, then $J(f) \cap \mathbb{C}^*$ has either one or infinitely many components (in [BD00], it was remarked that, in the latter case, there are uncountably many components). This implies that, if the closure of $J(f)$ in $\hat{\mathbb{C}}$ has two components, then $J(f) \cap \mathbb{C}^*$ has infinitely many components. Thus, one of the following three cases holds:

- both $J(f) \cap \mathbb{C}^*$ and its closure in $\hat{\mathbb{C}}$ are connected;
- $J(f) \cap \mathbb{C}^*$ has infinitely many components and its closure in $\hat{\mathbb{C}}$ is connected;
- $J(f) \cap \mathbb{C}^*$ has infinitely many components and its closure in $\hat{\mathbb{C}}$ has two components.

Note that we write $J(f) \cap \mathbb{C}^*$ because in the case that f is a transcendental entire function, $J(f)$ may contain the origin. Baker and Domínguez [BD98] also gave examples of functions in each of these three cases.

In terms of measure, Kotus [Kot87] proved that, under certain conditions, the Julia set of holomorphic self-maps of $\mathbb{C}^*$ has Lebesgue measure zero. In [Fan93], Fang showed that, for maps of the form $f(z) = z^n \exp(z^p + 1/z^q)$, $n \in \mathbb{Z}$, $p, q \geqslant 1$, the Julia set $J(f)$ has positive measure.

Very little work has been carried out on the escaping set of a general transcendental self-map of $\mathbb{C}^*$, although Fang [Fan98] introduced the following subsets of $I(f)$

$$I_0(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to 0 \text{ as } n \to \infty\},$$

$$I_\infty(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to \infty \text{ as } n \to \infty\},$$

and showed that they satisfy the analogues of Eremenko's properties, namely

$$I_0(f) \cap J(f) \neq \emptyset, \quad I_\infty(f) \cap J(f) \neq \emptyset, \quad J(f) = \partial I_0(f) = \partial I_\infty(f).$$

For this, Fang used Wiman-Valiron theory in the same way that Eremenko did for the entire case.

1.4 STRUCTURE OF THE THESIS

The escaping set of transcendental entire functions has been widely studied in recent years. However, there were very few results on the escaping set of transcendental self-maps of $\mathbb{C}^*$ and they only concerned points in the subsets $I_0(f)$ and $I_\infty(f)$ of $I(f)$, which consists of points that accumulate at $\{0, \infty\}$ in any possible way. The goal of this thesis is to study the escaping set of transcendental self-maps of $\mathbb{C}^*$ in greater detail and extend the recent research on the escaping set of transcendental entire functions to this class of functions.

In Chapter 2, we introduce the notion of *essential itinerary* of a point $z \in I(f)$ for a transcendental self-map f of $\mathbb{C}^*$, which is a sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$ that describes how the orbit of z accumulates to $\{0, \infty\}$. Then, for every sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, we define a completely invariant set $I_e(f) \subseteq I(f)$ that consists of all the points whose essential itinerary is, eventually, $\sigma^n(e)$ for some $n \in \mathbb{N}_0$, where σ is the Bernoulli shift map. Any two such sets are either equal or disjoint. Thus, we obtain a partition of $I(f)$ into uncountably many disjoint sets of the form $I_e(f)$ for some $e \in \{0,\infty\}^{\mathbb{N}_0}$. Note that the sets $I_0(f)$ and $I_\infty(f)$ correspond to the cases where e is the constant sequence 0 and ∞, respectively.

We prove the analogues of Eremenko's properties for each of the sets $I_e(f)$, $e \in \{0,\infty\}^{\mathbb{N}_0}$; namely, we show that, for each sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, $I_e(f) \cap J(f) \neq \emptyset$, $J(f) = \partial I_e(f)$ and all the components of the set $\overline{I_e(f)}$ are unbounded in $\mathbb{C}^*$. Note that, in particular, this means that there is an uncountable collection of disjoint sets whose boundary is the Julia set.

To that end, for each sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, we introduce the fast escaping set with essential itinerary e of a transcendental self-map f of $\mathbb{C}^*$, $A_e(f) \subseteq I_e(f)$, combining the iterates of the maximum and minimum modulus functions. We also adapt the construction of annular itineraries from [RS15], which allows us to construct fast escaping points, and also points that have almost every admissible itinerary with respect to an annular partition $\{A_n\}_{n\in\mathbb{Z}}$ defined using the iterates of the maximum *and* minimum modulus functions.

In Chapter 3, we focus on the escaping points in the Julia sets of transcendental self-maps of $\mathbb{C}^*$. We introduce the class $\mathcal{B}^*$ of transcendental self-maps of $\mathbb{C}^*$ of bounded type, and prove that, for functions $f \in \mathcal{B}^*$, we have $I(f) \subseteq J(f)$. We show that finite order functions in $\mathbb{C}^*$ are of the form $f(z) = z^n \exp(P(z) + Q(1/z))$ where $n \in \mathbb{Z}$ and P, Q are polynomials, and hence belong to the class $\mathcal{B}^*$. We also show that if $g, h \in \mathcal{B}$, the Eremenko-Lyubich class of transcendental entire functions, then the function $f(z) = \exp(g(z) + h(1/z))$ is in $\mathcal{B}^*$. We adapt the techniques from [RRRS11] to prove that, for finite compositions of transcendental self-maps of $\mathbb{C}^*$ of finite order, every point in $I(f)$ can be joined to zero or infinity by a ray tail (this generalises the work done in my masters thesis [Mar11] which relates to the special subsets $I_0(f)$ and $I_\infty(f)$ of $I(f)$). In particular, if a transcendental entire function is the lift of such a function, it follows that points whose real parts escape can be joined to infinity by a ray tail; note that these functions are not in the class $\mathcal{B}$. We also prove that all periodic external addresses correspond to a non-empty dynamic ray that lands. Finally, we show that for every sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, the set $I_e(f)$ contains a Cantor bouquet and, in particular, uncountably many ray tails.

In Chapter 4, we focus on the escaping points in the Fatou set of transcendental self-maps of $\mathbb{C}^*$. We provide the first explicit examples of transcendental self-maps of the punctured plane with a wandering domain and also with a Baker domain. In order to prove that our example has a wandering domain, we prove a general result which implies that a function has a bounded wandering domain and is of independent interest. For every sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, we use approximation theory to construct transcendental self-maps of $\mathbb{C}^*$ with wandering domains and, if e is periodic, with Baker domains in $I_e(f)$. We also construct transcendental entire and meromorphic functions that are holomorphic self-maps of $\mathbb{C}^*$ and have several types of escaping Fatou components.

In the first paper concerning the iteration of holomorphic self-maps of $\mathbb{C}^*$, Rådström [Råd53] described his goal as follows:

> *The theory of iteration developed by Fatou and Julia is concerned with rational and entire functions. What is the most general class of analytic functions to which the main results of this theory can be extended?*

In Chapter 5, we describe the works of Herring [Her95] and Bolsch [Bol97] on the iteration of functions that are holomorphic outside a small set of (generalised) essential singularities that are no longer isolated. For instance, these classes of functions include the iterates of transcendental meromorphic functions, that have infinite (countable) sets of essential singularities, like $f(z) = \exp(\tan z)$. Bolsch did not study the escaping set of such functions, and Herring only considered the subsets of the escaping set $I(f, \alpha)$ consisting of the points that accumulate at a given essential singularity α of f. The author regards this thesis as a step towards understanding the escaping set of more general functions, and plans to work on the escaping set of these classes of functions in the future.

In this direction, recently Nicks and Sixsmith [NS16] have adapted our definition of the fast escaping set in $\mathbb{C}^*$ to study the iteration of quasiregular functions of punctured space, that is, quasiregular functions $f : \mathbb{R}^d \setminus S \to \mathbb{R}^d \setminus S$ where $d > 2$ and $S \subseteq \mathbb{R}^d \cup \{\infty\}$ is a finite set that coincides with the set of essential singularities of f.

2

THE ESCAPING SET

In this chapter we study the structure of the escaping set of transcendental self-maps of $\mathbb{C}^*$. We introduce the notion of essential itinerary and prove the analogues of Eremenko's properties. In particular, we show that there is an uncountable collection of disjoint subsets of the escaping set each of which has the Julia set as its boundary. We define the fast escaping set for this class of functions by combining the iterates of the maximum and minimum modulus functions. We also use the maximum and minimum modulus functions to define an annular partition of $\mathbb{C}^*$ and then construct points with several types of annular itineraries with respect to that partition, including fast escaping points but also arbitrarily slowly escaping points.

2.1 INTRODUCTION AND MAIN RESULTS

Recall that, for transcendental self-maps of $\mathbb{C}^*$, we define the escaping set by

$$I(f) := \{z \in \mathbb{C}^* \ : \ \omega(z, f) \subseteq \{0, \infty\}\},$$

where $\omega(z, f) := \bigcap_{n\in\mathbb{N}} \overline{\{f^k(z) \ : \ k \geqslant n\}}$ and the closure is taken in $\hat{\mathbb{C}}$. The set $I(f)$ contains points that escape to zero as well as to infinity, defined as follows:

$$I_0(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to 0 \text{ as } n \to \infty\},$$
$$I_\infty(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The set $I(f)$ also contains points that escape from $\mathbb{C}^*$ by *jumping* infinitely many times between a neighbourhood of zero and a neighbourhood of infinity. The sets $I_0(f)$ and $I_\infty(f)$ were studied by Fang [Fan98] who proved the analogues of Eremenko's properties (I1) and (I2) described in Section 1.2, namely that

$$I_0(f) \cap J(f) \neq \emptyset, \quad I_\infty(f) \cap J(f) \neq \emptyset \quad \text{and} \quad J(f) = \partial I_0(f) = \partial I_\infty(f),$$

by using Wiman-Valiron theory in the way that Eremenko did for the entire case. But the full set of escaping points $I(f)$ has not been previously studied.

To classify the various types of escaping orbits we introduce the following concept.

Definition 2.1 (Essential itinerary)**.** Let f be a transcendental self-map of $\mathbb{C}^*$. We define the *essential itinerary* of a point $z \in I(f)$ to be the symbol sequence $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ such that

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

for all $n \in \mathbb{N}$.

We now introduce the set of points that escape with a particular essential itinerary.

Definition 2.2 (Escaping set)**.** For each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, the set of escaping points whose essential itinerary is *exactly* e,

$$I_e^{0,0} := \{z \in I(f) \ : \ \forall n \geqslant 0,\ |f^n(z)| > 1 \Leftrightarrow e_n = \infty\},$$

and, for $\ell, k \in \mathbb{N}_0$, we define

$$I_e^{-\ell,k} := \{z \in I(f) : \forall n \geqslant 0,\ |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\} = f^{-\ell}\big(I^{0,0}_{\sigma^k(e)}(f)\big),$$

where σ denotes the Bernoulli shift map. Finally, for $e \in \{0, \infty\}^{\mathbb{N}_0}$, we denote by $I_e(f)$ the set of escaping points whose essential itinerary is, *eventually, a shift of* e,

$$I_e(f) := \{z \in I(f) \ : \ \exists \ell, k \in \mathbb{N}_0,\ \forall n \geqslant 0,\ |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\},$$

or, equivalently,

$$I_e(f) := \bigcup_{\ell \in \mathbb{N}_0} \bigcup_{k \in \mathbb{N}_0} I_e^{-\ell,k}(f) = \bigcup_{\ell \in \mathbb{N}_0} \bigcup_{k \in \mathbb{N}_0} f^{-\ell}\big(I^{0,0}_{\sigma^k(e)}(f)\big).$$

We introduce a notion of *fast escaping set* related to an essential itinerary e, defined using the iterates of the maximum and minimum modulus functions

$$M(r,f) := \max_{|z|=r} |f(z)| < +\infty \quad \text{and} \quad m(r,f) := \min_{|z|=r} |f(z)| > 0,$$

which are defined for $r > 0$.

Definition 2.3 (Fast escaping set)**.** Let f be a transcendental self-map of $\mathbb{C}^*$. We define the *fast escaping set with respect to the essential itinerary* $e = (e_n) \in \{0,\infty\}^{\mathbb{N}_0}$, $A_e(f)$, as follows. First, let $R > 0$ be sufficiently large so that the sequence (R_n) defined by $R_0 := R$, if $e_0 = \infty$, or $R_0 := 1/R$, if $e_0 = 0$, and, for $n > 0$,

- $R_n := m(R_{n-1})$, if $e_n = 0$,
- $R_n := M(R_{n-1})$, if $e_n = \infty$,

accumulates to $\{0,\infty\}$. Then, $A_e^{-\ell,0}(f,R)$, $\ell \in \mathbb{Z}$, is defined to be the set of $z \in \mathbb{C}^*$ such that

- $|f^{n+\ell}(z)| \leqslant R_n$, if $e_n = 0$,
- $|f^{n+\ell}(z)| \geqslant R_n$, if $e_n = \infty$,

for all $n \in \mathbb{N}_0$ such that $n + \ell \in \mathbb{N}$, where $R > 0$. For $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, we put $A_e^{-\ell,k}(f,R) := A_{\sigma^k(e)}^{-\ell,0}(f,R) \subseteq I_e^{-\ell,k}(f)$. Finally, we define

$$A_e(f) := \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R).$$

We denote by $A(f)$ the *fast escaping set* of f, that is, the set of all points that are fast escaping with respect to some essential itinerary.

The sets $A_e(f)$, $e \in \{0,\infty\}^{\mathbb{N}_0}$, and $A(f)$ are independent of the value of $R > 0$ used to define them provided that R is large enough, and $A_e(f) \subseteq I_e(f)$ (see Lemmas 2.14 and 2.16).

Observe that if $f(z) = z^n \exp(g(z) + h(1/z))$ with $n \in \mathbb{Z}$ and g, h non-constant entire functions, then the behaviour of f in a neighbourhood of infinity depends mainly on that of the entire function g while the behaviour near zero depends mainly on that of h.

We begin by proving an analogue of property (I1), namely that $I_e(f) \cap J(f)$ and indeed $A_e(f) \cap J(f)$ are non-empty for *any* essential itinerary e. We follow the approach of Rippon and Stallard in [RS15] where they proved the existence of points escaping to infinity at different rates by constructing points with different annular itineraries.

Theorem 2.4. *Let f be a transcendental self-map of $\mathbb{C}^*$. For each sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, we have $A_e(f) \cap J(f) \neq \emptyset$ and hence $I_e(f) \cap J(f) \neq \emptyset$.*

Our notation for annular itineraries is as follows. Let $R_+ > 0$ and $R_- > 0$ be, respectively, large enough and small enough such that, for all $r > R_+$, $M(r) > r$ and, for all $0 < r < R_-$, we have $m(r) < r$. Then define

$$A_0 := \overline{A}(R_-, R_+) = \{z \in \mathbb{C}^* \ : \ R_- \leqslant |z| \leqslant R_+\}$$

and the sequences of annuli

$$\begin{aligned} A_n &:= \{z \in \mathbb{C}^* \ : \ M^{n-1}(R_+) < |z| \leqslant M^n(R_+)\}, &&\text{for } n > 0; \\ A_n &:= \{z \in \mathbb{C}^* \ : \ m^{-n}(R_-) \leqslant |z| < m^{-n-1}(R_-)\}, &&\text{for } n < 0. \end{aligned}$$

Each point $z \in I(f)$ has an associated *annular itinerary* $(s_n) \in \mathbb{Z}^{\mathbb{N}_0}$ with respect to the partition $\{A_n\}_{n\in\mathbb{Z}}$ such that $f^n(z) \in A_{s_n}$ for all $n \in \mathbb{N}_0$. We prove a covering result (see Theorem 2.18) which allows us to construct orbits with certain annular itineraries, including the ones listed in Theorem 2.6 below.

Remark 2.5. In this thesis we deal with two kinds of itineraries for escaping points that should not be confused: essential itineraries $(e_n) \in \{0,\infty\}^{\mathbb{N}_0}$, which describe how an escaping point accumulates to the two essential singularities, and annular itineraries $(s_n) \in \mathbb{Z}^{\mathbb{N}_0}$, which depend on the partition $\{A_n\}_{n\in\mathbb{Z}}$. For large values of n, the symbols $e_n = 0$ and $e_n = \infty$ correspond, respectively, to negative and positive terms s_n in the annular itinerary.

Theorem 2.6. *Let f be a transcendental self-map of $\mathbb{C}^*$. Given an annular partition $\{A_n\}_{n\in\mathbb{Z}}$ defined as above with $R_+, 1/R_-$ sufficiently large, we can construct points with the following itineraries:*

- *fast escaping itineraries;*
- *periodic itineraries;*
- *bounded itineraries (uncountably many);*
- *unbounded non-escaping itineraries (uncountably many);*
- *arbitrarily slowly escaping itineraries.*

Note that our proof uses a different annular covering lemma to those used in [RS15] and, in this setting, we are able to avoid the exceptional sets which feature in [RS15, Theorem 1.1 and Theorem 1.2].

We now state a result in the spirit of property (I2) but for *any* essential itinerary e. For the special cases of $I_0(f)$ and $I_\infty(f)$, this is due to Fang and it also follows from the results in [BDH01].

Theorem 2.7. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$. *For each sequence* $e \in \{0, \infty\}^{\mathbb{N}_0}$, *we have* $J(f) = \partial A_e(f) = \partial I_e(f)$. *Also* $J(f) = \partial A(f) = \partial I(f)$.

Since there are uncountably many non-equivalent essential itineraries (see Remark 2.13(ii)), this means, in particular, that there is an uncountable collection of disjoint sets, each of which has the Julia set as its boundary.

We also prove the analogue of property (I3) for any essential itinerary. When we say that a set X is *unbounded* in $\mathbb{C}^*$, we mean that $\widehat{X} \cap \{0, \infty\} \neq \emptyset$, where $\widehat{X}$ is the closure of X in $\hat{\mathbb{C}}$.

Theorem 2.8. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$. *For each sequence* $e \in \{0, \infty\}^{\mathbb{N}_0}$, *the connected components of* $\overline{I_e(f)}$ *are unbounded in* $\mathbb{C}^*$, *and hence the connected components of* $\overline{I(f)}$ *are unbounded in* $\mathbb{C}^*$.

Finally we show that, as for transcendental entire functions, the components of $A(f)$ are all unbounded.

Theorem 2.9. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$. *For each sequence* $e \in \{0, \infty\}^{\mathbb{N}_0}$, *the connected components of* $A_e(f)$ *are unbounded, and hence the connected components of* $A(f)$ *are unbounded.*

Structure of the chapter. In Section 2.2 we prove the basic properties of $M(r)$ and $m(r)$ that we are going to need later. The discussion about the notions of essential itinerary and the fast escaping set is in Section 2.3. Section 2.4 is devoted to the construction of the annular itineraries and the proof of Theorem 2.6. The main result in this section, Theorem 2.18, in fact allows you to construct many more types of orbits than the ones listed in the statement of Theorem 2.6. In Section 2.5 we prove Theorems 2.4, 2.7 and 2.8 which are the analogues of Eremenko's properties (I1), (I2) and (I3) in $\mathbb{C}^*$. In doing so we also show that the components of the fast escaping set are unbounded (see Theorem 2.9), and that if a Fatou component U intersects the fast escaping set $A(f)$ then $\overline{U} \subseteq A(f)$ (see Theorem 2.21).

2.2 THE MAXIMUM AND MINIMUM MODULUS FUNCTIONS

Before proving the annular covering results, we need some basic properties of the maximum and minimum modulus functions for transcendental self-maps of $\mathbb{C}^*$. Note that we will not usually make explicit the dependence on f and we will just write $M(r)$ and $m(r)$. As a consequence of the maximum modulus principle, both $M(r)$ and $m(r)$ are unimodal functions. In the following lemma we summarise their main properties. Throughout this section we will only prove the statements for $M(r)$ when $r \to +\infty$, and the other three statements for $M(r)$ when $r \to 0$ and for $m(r)$ when $r \to +\infty$ and $r \to 0$ can be deduced from these by using the fact that if $\check{f}(z) = f(1/z)$ then

$$M(r, f) = M(1/r, \check{f}) = \frac{1}{m(r, 1/f)} = \frac{1}{m(1/r, 1/\check{f})}.$$

Lemma 2.10. *Let f be a transcendental self-map of $\mathbb{C}^*$. The functions $M(r)$ and $m(r)$ satisfy the following properties:*

(i) $\dfrac{\log M(r)}{\log r} \to +\infty,\ \dfrac{\log m(r)}{\log r} \to -\infty$ *as* $r \to +\infty$, *and*

$$\frac{\log M(r)}{\log r} \to -\infty,\ \frac{\log m(r)}{\log r} \to +\infty \text{ as } r \to 0;$$

(ii) $\log M(r)$ *and* $-\log m(r)$ *are convex functions of* $\log r$*;*

(iii) there exists $R^\infty = R^\infty(f) > 0$ *such that*

$$M(r^k) \geqslant M(r)^k,\ m(r^k) \leqslant m(r)^k \text{ for every } r \geqslant R^\infty,\ k > 1,$$

and there exists $R^0 = R^0(f) > 0$ *such that*

$$M(r^k) \geqslant M(r)^k,\ m(r^k) \leqslant m(r)^k \text{ for every } r \leqslant R^0,\ k > 1;$$

(iv) for $k > 1$, $\dfrac{M(kr)}{M(r)} \to +\infty,\ \dfrac{m(kr)}{m(r)} \to 0$ *as* $r \to +\infty$, *and*

$$\frac{M(kr)}{M(r)} \to 0,\ \frac{m(kr)}{m(r)} \to +\infty \text{ as } r \to 0.$$

Proof. (i) This property follows from the fact that

$$\frac{\log M(r)}{\log r} \to +\infty \quad \text{as } r \to +\infty$$

for transcendental entire functions (see [Lev96, Theorem 1 on p. 3]) using that $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$, where $n \in \mathbb{Z}$ and g, h are non-constant entire functions (see (1.1)), so

$$\lim_{r\to+\infty} \frac{\log M(r,f)}{\log r} = \lim_{r\to+\infty} \frac{\log M(r,f_\infty)}{\log r} = +\infty,$$

where $f_\infty(z) = z^n \exp g(z)$.

(ii) This means that $\phi(t) = \log M(\exp t)$ is a convex function of t and the property is usually referred to as the Hadamard three circles theorem, see [Ahl53]. Observe that in the hypothesis of that theorem you only need that the function is analytic in an annulus $r_1 < |z| < r_2$ and it therefore applies to holomorphic self-maps of $\mathbb{C}^*$.

(iii) See [RS09, Lemma 2.2] or [BRS13, Theorem 2.2] for the analogous result for transcendental entire functions. We reproduce the proof here for completeness.

Let $\phi(t) = \log M(\exp t)$. By property (i), $\phi(t)/t \to +\infty$ as $t \to +\infty$, so we can take $t_1 \geqslant t_0 > 0$ large enough that

$$\phi(t_0) > 0 \quad \text{and} \quad \frac{\phi(t)}{t} \geqslant \frac{\phi(t_0)}{t_0} \quad \text{for } t \geqslant t_1.$$

Let ϕ' denote the right derivative of ϕ. Then, by property (ii) and the previous inequality,

$$\phi'(t) \geqslant \frac{\phi(t) - \phi(t_0)}{t - t_0} \geqslant \frac{\phi(t)}{t} \quad \text{for } t \geqslant t_1.$$

Hence $\phi(t)/t$ is an increasing function for $t \geqslant t_1$. Thus, if $k > 1$, then

$$\frac{\phi(kt)}{kt} \geqslant \frac{\phi(t)}{t}, \text{ that is, } \phi(kt) \geqslant k\phi(t),$$

for $t \geqslant t_1$. Taking exponentials on both sides we get the result, with $R_\infty = \exp t_1$.

(iv) For every value of $r > 1$ we can write $kr = r^c$, where

$$c = c(r) = \frac{\log k + \log r}{\log r} > 1.$$

By property (iii), for r large enough,

$$\frac{M(kr)}{M(r)} = \frac{M(r^c)}{M(r)} \geqslant \frac{M(r)^c}{M(r)} = M(r)^{c-1}$$

and then, using property (i),

$$\log\left(M(r)^{c-1}\right) = (c-1)\log M(r) = \frac{\log k}{\log r}\log M(r) \to +\infty \text{ as } r \to +\infty,$$

so

$$\frac{M(kr)}{M(r)} \to +\infty \quad \text{as } r \to +\infty. \qquad \blacksquare$$

The following result compares the iterates of $M(r)$ and $m(r)$ with those of their 'relaxed' versions $\mu(r) = \varepsilon M(r)$ and $\nu(r) = m(r)/\varepsilon$, where $0 < \varepsilon < 1$. The analogous property for entire functions was used by Rippon and Stallard in [RS12, Theorem 2.9].

Lemma 2.11. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$, *and define* $\mu(r) = \varepsilon M(r)$ *and* $\nu(r) = m(r)/\varepsilon$, *where* $0 < \varepsilon < 1$. *Then there exists* $R_1(f, \varepsilon) > 0$ *such that, for* $r \geqslant R_1(f, \varepsilon)$,

$$\mu^n(r) \geqslant M^n(\varepsilon r) \quad \textit{and} \quad \nu^n(r) \leqslant m^n(\varepsilon r) \quad \textit{for } n > 0,$$

and, for $0 < r \leqslant 1/R_1(f, \varepsilon)$,

$$\mu^n(r) \geqslant M^n(r/\varepsilon) \quad \textit{and} \quad \nu^n(r) \leqslant m^n(r/\varepsilon) \quad \textit{for } n > 0.$$

Proof. Let R be large enough that $M(\varepsilon r) \geqslant \varepsilon r$ for all $r \geqslant R$. By property (iv) in Lemma 2.10, with $k = 1/\varepsilon$, there is $R' \geqslant R$ such that,

$$\frac{M(r)}{M(\varepsilon r)} \geqslant \frac{1}{\varepsilon^2} \quad \text{for } r \geqslant R',$$

and therefore

$$\mu(r) = \varepsilon M(r) \geqslant \frac{1}{\varepsilon} M(\varepsilon r) \geqslant r \quad \text{for } r \geqslant R'.$$

Hence, $\mu^n(r) \geqslant M^n(\varepsilon r)$ for all $n \in \mathbb{N}$ and $r \geqslant R'$. If $R'' > 0$ is the constant required for the corresponding inequality with $m(r)$ and $\nu(r)$, then we define $S := \max\{R', R''\}$. If $S' > 0$ is the constant such that the second pair of inequalities hold for $0 < r < S'$, then we put $R_1(f, \varepsilon) := \max\{S, 1/S'\}$. $\blacksquare$

Finally let us prove a property of $M(r)$ and $m(r)$ that will be used later in the construction of the annular itineraries.

Lemma 2.12. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$, *and define* $\mu(r) = \varepsilon M(r)$ *and* $\nu(r) = m(r)/\varepsilon$, *where* $0 < \varepsilon < 1$. *Then there exists* $R_2(f, \varepsilon) > 0$ *such that, for* $r \geqslant R_2(f, \varepsilon)$,

$$M^{n-1}(r) < \varepsilon\mu^n(r) \quad \textit{for } n > 0,$$

and, for $0 < r \leqslant 1/R_2(f, \varepsilon)$,

$$m^{n-1}(r) > \nu^n(r)/\varepsilon \quad \textit{for } n > 0.$$

Proof. Consider $\tilde{\mu}(r) = \varepsilon^2 M(r)$ and let $R_1(f, \varepsilon^2) > 0$ be the constant defined in Lemma 2.11. Then

$$\tilde{\mu}^n(r) \geqslant M^n(\varepsilon^2 r)$$

for all $n \in \mathbb{N}$ and $r \geqslant R_1(f, \varepsilon^2)$. Now let $R > R_1(f, \varepsilon^2)$ be large enough that $r \leqslant M(\varepsilon^2 r)$ for all $r \geqslant R$. Then, applying M^{n-1} to both sides of the inequality $r \leqslant M(\varepsilon^2 r)$, we get

$$M^{n-1}(r) \leqslant M^n(\varepsilon^2 r) \leqslant \tilde{\mu}^n(r)$$

for $r \geqslant R$. Hence,

$$M^{n-1}(r) \leqslant \tilde{\mu}^n(r) = \varepsilon^2 M\big(\tilde{\mu}^{n-1}(r)\big) < \varepsilon^2 M\big(\mu^{n-1}(r)\big) = \varepsilon\mu^n(r)$$

for all $n \in \mathbb{N}$ and $r \geqslant R$. If $R' > 0$ is the constant required for the corresponding inequality with $m(r)$ and $\nu(r)$, then the required result holds with $R_2(f, \varepsilon) := \max\{R, 1/R'\}$. ∎

2.3 THE ESCAPING AND FAST ESCAPING SETS

In this section we discuss some basic properties of the escaping and fast escaping sets of transcendental self-maps of $\mathbb{C}^*$. Recall that in Definition 2.1 we defined the *essential itinerary* of an escaping point $z \in I(f)$ to be the symbol sequence $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ such that

$$e_n = \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

and $I_e(f)$ denotes the set of points whose essential itinerary is, eventually, a shift of e, that is,

$$I_e(f) := \{z \in I(f) \,:\, \exists \ell, k \in \mathbb{N}_0,\ \forall n \in \mathbb{N}_0,\ |f^{n+\ell}(z)| > 1 \;\Leftrightarrow\; e_{n+k} = \infty\}.$$

Remark 2.13.

(i) Observe that we used the unit circle to define the boundaries of neighbourhoods of zero and infinity but we could have used any circle $\{z : |z| = R\}$ with $R > 0$, because the orbits of escaping points are eventually as close as we want to the essential singularities.

(ii) For $e \neq e'$, the sets $I_e(f)$ and $I_{e'}(f)$ are either equal or disjoint. In fact, we have $I_e(f) = I_{e'}(f)$ if and only if $\sigma^m(e) = \sigma^n(e')$ for some $m, n \in \mathbb{N}_0$, where σ denotes the Bernoulli shift map. In this case we say that e is *equivalent* to e' and write $e \cong e'$. However, it is easy to see that there are uncountably many non-equivalent essential itineraries.

(iii) We use the notation $\overline{e_0 e_1 \ldots e_{p-1}}$, $p \in \mathbb{N}_0$, to denote the periodic sequence of period p which consists of $e_0 e_1 \ldots e_{p-1}$ repeated infinitely often. This notation will be used for annular itineraries as well.

Recall that in Definition 2.3 we defined the *fast escaping set*, $A(f)$, by iterating a combination of $M(r)$ and $m(r)$ on $R_0 > 0$ following an essential itinerary e. In order for this set to be well defined, we first need to guarantee that the sequence (R_n) escapes to $\{0, +\infty\}$ provided that R_0 is sufficiently large, if $e_0 = \infty$, or sufficiently small, if $e_0 = 0$.

Lemma 2.14. *Let f be a transcendental self-map of $\mathbb{C}^*$. There is $R(f) > 0$ so large that, for every $e \in \{0, \infty\}^{\mathbb{N}}$, if the sequence (R_n) is as in Definition 2.3, where $R > R(f)$, then*

(i) $M(r) > r^2$ and $1/m(r) > r^2$, if $r > R(f)$, and $1/M(r) < r^2$ and $m(r) < r^2$, if $0 < r < 1/R(f)$, and hence $R_n \to \{0, +\infty\}$ as $n \to \infty$;

(ii) if $R' > R$ and (R'_n) is the sequence defined using R' as in Definition 2.3, then, for all $n \in \mathbb{N}_0$, $R'_n > R_n$, if $e_n = \infty$, and $R'_n < R_n$, if $e_n = 0$.

It follows that $A_e(f) \subseteq I_e(f)$.

Proof. Since $M(r)$ and $1/m(r)$ grow faster than any power of r (see Lemma 2.10(i)), we can take $R(f) > 0$ large enough that (i) holds.

Moreover, by Lemma 2.10(ii), we can choose $R(f)$ sufficiently large so that, in addition,

- the function $M(r)$ is monotonically increasing on $(R(f), +\infty)$ and monotonically decreasing on $(0, 1/R(f))$;
- the function $m(r)$ is monotonically decreasing on $(R(f), +\infty)$ and monotonically increasing on $(0, 1/R(f))$.

Thus, if $R' > R$, the sequence (R'_n) will *beat* the sequence (R_n), that is, for all $n \in \mathbb{N}_0$, $R'_n > R_n$, if $e_n = \infty$, and $R'_n < R_n$, if $e_n = 0$. ■

Let $R > R(f)$, where $R(f)$ is the constant from Lemma 2.14. The *fast escaping set with respect to the essential itinerary* $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ was defined in Section 2.1 to be

$$A_e(f) := \bigcup_{\ell \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}_0} A_e^{-\ell,k}(f, R),$$

where the set $A_e^{-\ell,k}(f, R)$, called a *level* of $A_e(f)$, consists of the points $z \in \mathbb{C}^*$ such that,

- $|f^{n+\ell}(z)| \leqslant R_n$, if $e_{n+k} = 0$,
- $|f^{n+\ell}(z)| \geqslant R_n$, if $e_{n+k} = \infty$,

for all $n \in \mathbb{N}_0$ such that $n + \ell \in \mathbb{N}_0$. For $A_e^{-\ell,k}(f, R)$, the sequence (R_n) is defined by $R_0 := R$, if $e_k = \infty$, and $R_0 := 1/R > m(R)$, if $e_k = 0$, and, for $n > 0$,

- $R_n := m(R_{n-1})$, if $e_{n+k} = 0$,
- $R_n := M(R_{n-1})$, if $e_{n+k} = \infty$.

Observe that the sets $A_e^{-\ell,k}(f, R) = A_{\sigma^k(e)}^{-\ell,0}(f, R)$ are closed. Also, note that, in these definitions, we could have chosen $0 < R < 1/R(f)$ and then taken $R_0 = R$, if $e_k = 0$, and $R_0 = 1/R < M(R)$, if $e_k = \infty$.

Lemma 2.15. *Let f be a transcendental self-map of $\mathbb{C}^*$, and let $e \in \{0, \infty\}^{\mathbb{N}_0}$, $R > R(f)$, where $R(f)$ is as defined in Lemma 2.14, $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Then we have $A_e^{-\ell,k}(f, R) \subseteq A_e^{-\ell-1,k+1}(f, R)$ and hence*

$$A_e(f) = \bigcup_{\ell \in \mathbb{N}_0} \bigcup_{k \in \mathbb{N}_0} A_e^{-\ell,k}(f, R).$$

Proof. Let (R_n) be the sequence defined by $R_0 := R$, if $e_k = \infty$, and $R_0 := 1/R$, if $e_k = 0$, and, for $n > 0$,

- $R_n := m(R_{n-1})$, if $e_{n+k} = 0$,
- $R_n := M(R_{n-1})$, if $e_{n+k} = \infty$,

and let $(\widetilde{R_n})$ be the sequence defined by $\widetilde{R_0} := R$, if $e_{k+1} = \infty$, and $\widetilde{R_0} := 1/R$, if $e_{k+1} = 0$, and, for $n > 0$,

- $\widetilde{R_n} := m(\widetilde{R_{n-1}})$, if $e_{n+k+1} = 0$,
- $\widetilde{R_n} := M(\widetilde{R_{n-1}})$, if $e_{n+k+1} = \infty$.

By Lemma 2.14(i) and (ii), $R_{n+1} > \widetilde{R_n} > R(f)$, if $e_{n+k+1} = \infty$, and $R_{n+1} < \widetilde{R_n} < 1/R(f)$, if $e_{n+k+1} = 0$, for all $n \in \mathbb{N}_0$.

Suppose that $z_0 \in A_e^{-\ell,k}(f, R)$; then

- $|f^{n+\ell}(z_0)| \leqslant R_n$, if $e_{n+k} = 0$,
- $|f^{n+\ell}(z_0)| \geqslant R_n$, if $e_{n+k} = \infty$,

for all $n \in \mathbb{N}_0$ such that $n + \ell \in \mathbb{N}_0$, and therefore

- $|f^{n+\ell+1}(z_0)| \leqslant R_{n+1} < \widetilde{R_n}$, if $e_{n+k+1} = 0$,
- $|f^{n+\ell+1}(z_0)| \geqslant R_{n+1} > \widetilde{R_n}$, if $e_{n+k+1} = \infty$,

for all $n \in \mathbb{N}_0$ such that $n+\ell+1 \in \mathbb{N}_0$, and thus $z_0 \in A_e^{-\ell-1,k+1}(f, R)$. Observe that if $\ell < 0$, then points in both of the sets $A_e^{-\ell,k}(f, R)$ and $A_e^{-\ell-1,k+1}(f, R)$ must satisfy respectively the conditions above for each iterate $f^n(z_0)$, $n \in \mathbb{N}_0$. If $\ell \geqslant 0$, then points in $A_e^{-\ell,k}(f, R)$ must satisfy a condition on $f^\ell(z_0)$ while for points in $A_e^{-\ell-1,k+1}(f, R)$, the iterate $f^\ell(z_0)$ is arbitrary.

Finally, if $\ell < 0$, then $A_e^{-\ell,k}(f, R) \subseteq A_e^{0,k-\ell}(f, R)$ and, therefore, we can define $A_e(f)$ using only the level sets $A_e^{-\ell,k}(f, R)$ with $\ell \in \mathbb{N}_0$. ∎

The following lemma shows that the sets $A_e(f)$ (and hence also $A(f)$) are independent of R, as mentioned in Section 2.1, and completely invariant under f. The sets $A_e(f)$ are also invariant under shifts of e; that is, $A_{\sigma(e)}(f) = A_e(f)$.

Lemma 2.16. *Let f be a transcendental self-map of $\mathbb{C}^*$. For each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, the set $A_e(f)$ is completely invariant under f, shift invariant and independent of R, provided that $R > R(f)$, where $R(f)$ is the constant from Lemma 2.14. Hence the set $A(f)$ is completely invariant and independent of R.*

Proof. We first show that the set $A_e(f)$ is completely invariant under f, that is,

$$f\big(A_e(f)\big) \subseteq A_e(f) \quad \text{and} \quad f^{-1}\big(A_e(f)\big) \subseteq A_e(f).$$

The left-hand inclusion holds because $f\big(A_e^{-\ell,k}(f,R)\big) \subseteq A_e^{-\ell+1,k}(f,R)$, which follows easily from the definition of the levels of $A_e(f)$. To prove the right-hand inclusion, we note that, by Lemma 2.15, we can suppose that $\ell \in \mathbb{N}$ and in that case

$$f^{-1}\big(A_e^{-\ell,k}(f,R)\big) = A_e^{-\ell-1,k}(f,R).$$

We now show that the set $A_e(f)$ is shift invariant, that is, we prove that $A_{\sigma(e)}(f) = A_e(f)$. First,

$$\begin{aligned} A_{\sigma(e)}(f) &= \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_{\sigma(e)}^{-\ell,k}(f,R) = \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k+1}(f,R) \\ &= \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\geqslant 1} A_e^{-\ell,k}(f,R) \subseteq A_e(f). \end{aligned}$$

In the other direction, by Lemma 2.15,

$$A_e(f) = \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R) \subseteq \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell-1,k+1}(f,R) = A_{\sigma(e)}(f).$$

We give the details that $A_e(f)$ is independent of R for the case where there exists a sequence (n_k) such that $R_{n_k} \to +\infty$ as $k \to \infty$. Otherwise $R_n \to 0$ as $n \to \infty$ and the argument is similar.

Suppose that $R' > R > R(f)$ and let (R_n) and (R'_n) be the sequences given by Definition 2.3 starting with R and R', respectively. By Lemma 2.14, we have that $R'_n \to \{0, +\infty\}$ as $n \to \infty$ and $A_e^{-\ell,k}(f,R') \subseteq A_e^{-\ell,k}(f,R)$ for $\ell \in \mathbb{Z}$, $k \in \mathbb{N}_0$. Hence,

$$\bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R') \subseteq \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R).$$

In the other direction, we use the fact that we have assumed that there is a sequence (n_k) such that $R_{n_k} \to +\infty$ as $k \to \infty$. Let $m \in \mathbb{N}_0$

be such that $R_m > R'$. If $\tilde{e}$ is the symbol sequence e preceded by the string $e_0 \dots e_{m-1}$, then

$$A_e^{-\ell,k}(f,R') \supseteq A_e^{-\ell,k}(f,R_m) = A_{\tilde{e}}^{-\ell,k+m}(f,R_m) \supseteq A_{\tilde{e}}^{-(\ell-m),k}(f,R)$$

and, since $\sigma^m(\tilde{e}) = e$, by the shift invariance property, we have

$$\bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R') \supseteq \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_{\tilde{e}}^{-(\ell-m),k}(f,R) \supseteq \bigcup_{\ell\in\mathbb{Z}} \bigcup_{k\in\mathbb{N}_0} A_e^{-\ell,k}(f,R)$$

Therefore $A_e(f)$ is independent of the value of R used to define it. ■

Since $A_e(f)$ is shift invariant, if $e, e' \in \{0,\infty\}^{\mathbb{N}_0}$ and $A_{e'}(f) \cap I_e(f) \neq \emptyset$, then $A_{e'}(f) = A_e(f)$, by Remark 2.13(ii).

We will continue studying the dynamical and topological properties of $A(f)$ in Section 2.5.

2.4 ANNULAR ITINERARIES FOR $\mathbb{C}^*$

In this section, we study annular itineraries for our class of functions. By Lemma 2.10, there exist $R_+, R_- > 0$, respectively, large and small enough such that $M^n(R_+) \to +\infty$ and $m^n(R_-) \to 0$ as $n \to \infty$. We define $A_0 := \overline{A}(R_-, R_+)$ and

$$\begin{aligned} A_n &:= \{z \in \mathbb{C}^* \ : \ M^{n-1}(R_+) < |z| \leqslant M^n(R_+)\} && \text{for } n > 0, \\ A_n &:= \{z \in \mathbb{C}^* \ : \ m^{-n}(R_-) \leqslant |z| < m^{-n-1}(R_-)\} && \text{for } n < 0, \end{aligned}$$

so that $\{A_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{C}^*$. Each point $z \in I(f)$ has an associated *annular itinerary* $(s_n) \in \mathbb{Z}^{\mathbb{N}_0}$ such that $f^n(z) \in A_{s_n}$ for all $n \in \mathbb{N}_0$. Note that this sequence depends on the values R_- and R_+ used to define the partition. By construction, it follows from the maximum modulus principle that

$$\begin{aligned} s_{n+1} &\leqslant s_n + 1, && \text{if } s_n > 0, \\ s_{n+1} &\geqslant s_n - 1, && \text{if } s_n < 0. \end{aligned}$$

To create escaping orbits with certain types of annular itineraries we will use the following version of a well-known result.

Lemma 2.17. *Let $\{C_n\}_{n\in\mathbb{N}_0}$ be compact sets in $\mathbb{C}^*$ and $f : \mathbb{C}^* \to \mathbb{C}^*$ be a continuous function such that*

$$f(C_n) \supseteq C_{n+1} \quad \textit{for } n \in \mathbb{N}_0.$$

Then there exists ζ such that $f^n(\zeta) \in C_n$ for $n \in \mathbb{N}_0$.

In our construction, the compact sets $\{C_n\}_{n\in\mathbb{N}_0}$ will be compact annuli $B_n \subseteq A_n$ with some covering properties. More precisely, we will have that $B_n \subseteq \operatorname{int} A_n$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

Theorem 2.18. *Let f be a transcendental self-map of $\mathbb{C}^*$. If $\{A_n\}_{n\in\mathbb{Z}}$ is the set of annuli defined above, then there exists a sequence of closed annuli $\{B_n\}_{n\in\mathbb{Z}}$ such that $B_n \subseteq A_n$ for all $n \in \mathbb{Z}$, with the following covering properties:*

- *if $n > 0$, there exists an integer $k_n \leqslant 1$ such that $f(B_n) \supseteq B_k$ for $k_n \leqslant k \leqslant n+1$,*
- *if $n < 0$, there exists an integer $k_n \geqslant -1$ such that $f(B_n) \supseteq B_k$ for $n-1 \leqslant k \leqslant k_n$,*

and $|k_m| \geqslant |k_n|$ when $|m| > |n|$ and m, n have the same sign. Moreover, $|k_m| \to \infty$ as $|m| \to \infty$.

Note that the sequence (k_m) depends on the growth of the function $m(r)$ as $r \to +\infty$ (when $m > 0$) and on that of $M(r)$ as $r \to 0$ (when $m < 0$).

We compare Theorem 2.18 with the corresponding result for transcendental entire functions [RS15, Theorem 1.1]. In that setting, there is a subsequence (n_j) such that $f(B_{n_j}) \supseteq B_k$ for $0 \leqslant k \leqslant n_j + 1$ with at most one exception while, in our case, all $f(B_n)$ cover the other B_k with $0 < k \leqslant n+1$ if $n > 0$. Also, the proof in [RS15] is significantly more involved than ours due to the possible presence of zeros of the function and multiply connected Fatou components, and it requires the use of several new covering lemmas.

In order to prove Theorem 2.18, we use the following recent covering result due to Bergweiler, Rippon and Stallard (see [BRS13, Theorem 3.3]). Here $[z, w]_\Omega$ stands for the hyperbolic distance between the points z and w relative to Ω, where Ω is a hyperbolic domain; that is, Ω has at least two finite boundary points.

Lemma 2.19. *There exists an absolute constant* $\delta > 0$ *such that if* $R' > R$ *and* $f : A(R, R') \to \mathbb{C}^*$ *is analytic, then, for all* $z_1, z_2 \in A(R, R')$ *such that*

$$[z_1, z_2]_{A(R,R')} < \delta \quad \text{and} \quad |f(z_2)| \geqslant 2|f(z_1)|,$$

we have

$$f\big(A(R, R')\big) \supseteq \overline{A}\big(|f(z_1)|, |f(z_2)|\big).$$

Now we prove Theorem 2.18.

Proof of Theorem 2.18. If $A(\varepsilon) = A(\varepsilon, \frac{1}{\varepsilon})$, $0 < \varepsilon < 1$, and $C = \{z : |z| = 1\}$, then the hyperbolic length of C with respect to $A(\varepsilon)$ is

$$\ell_{A(\varepsilon)}(C) = \frac{\pi^2}{\log(1/\varepsilon)}$$

(see [BM07, Example 12.1]). Since the hyperbolic length is invariant under conformal transformations, the hyperbolic length of the circle $\{z : |z| = r\}$ with respect to $A(\varepsilon r, \frac{1}{\varepsilon} r)$ is also $-\pi^2/\log \varepsilon$. We choose $0 < \varepsilon < 1$ to be sufficiently small that

$$\frac{\pi^2}{\log(1/\varepsilon)} < \delta,$$

where δ is the absolute constant of Lemma 2.19.

Let $\mu(r) = \varepsilon M(r)$ and let $R_2(f, \varepsilon)$ be the constant in Lemma 2.12. Then we claim that there exists $R_0 = R_0(f, \varepsilon) > R_2(f, \varepsilon) > 0$ such that if $R_+ \geqslant R_0$, then

$$M^{n-1}(R_+) < \varepsilon\mu^n(R_+) < \mu^n(R_+) < \frac{1}{\varepsilon}\mu^n(R_+) < M^n(R_+) \qquad (2.1)$$

for all $n > 1$ (see Figure 3). Note that if $n = 1$ the first three inequalities in (2.1) hold while the last one becomes an equality. Indeed Lemma 2.12 ensures that the first inequality is satisfied. The two middle inequalities are clear because $0 < \varepsilon < 1$, and the last one is due to the fact that the function $\mu(r)$ is increasing for large values of r and $\mu^{n-1}(R_+) < M^{n-1}(R_+)$.

Let $B_0 := \overline{A_0}$ and, for $n > 0$, define

$$B_n := \overline{A}\left(\varepsilon\mu^n(R_+),\ \tfrac{1}{\varepsilon}\mu^n(R_+)\right) \subseteq A_n.$$

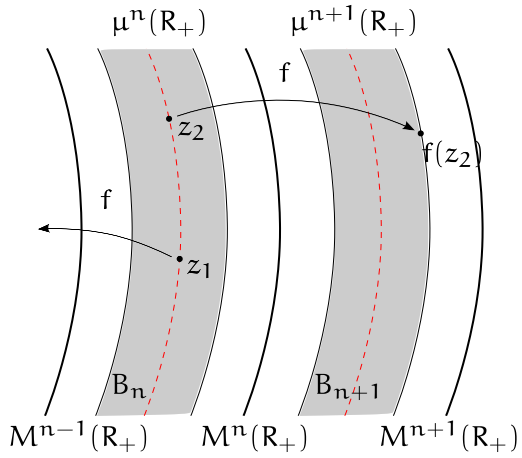


Figure 3: Construction in the proof of Theorem 2.18.

The same kind of argument can be used to construct $R_- > 0$ and annuli B_n for $n < 0$ in an analogous way, using the iterates of the function $\nu(r) = m(r)/\varepsilon$ with initial term $r = R_-$.

Take $z_1, z_2 \in B_n$ such that $|z_1| = |z_2| = \mu^n(R_+)$ and

$$|f(z_1)| = m\big(\mu^n(R_+)\big), \quad |f(z_2)| = M\big(\mu^n(R_+)\big).$$

This is possible by (2.1). Our choice of ε ensures that $[z_1, z_2]_{B_n} < \delta$ and, for R_+ large enough, the condition $|f(z_2)| \geqslant 2|f(z_1)|$ is trivially satisfied.

Finally, observe that if R_+ is large enough, we can make sure that, for every $n > 0$, $m\big(\mu^n(R_+)\big) < R_+$. Then Lemma 2.19 tells us that

$$\begin{aligned} f(B_n) &\supseteq \overline{A}\big(|f(z_1)|, |f(z_2)|\big) = \overline{A}\Big(m\big(\mu^n(R_+)\big), M\big(\mu^n(R_+)\big)\Big) \\ &\supseteq \overline{A}\left(R_+, \tfrac{1}{\varepsilon}\mu^{n+1}(R_+)\right) \supseteq \bigcup_{j=k_n}^{n+1} B_j \end{aligned}$$

with $k_n \leqslant 1$, as required for the case $n > 0$. The proof that B_n satisfies the corresponding covering properties for $n < 0$ is analogous. ∎

Remark 2.20. Note that B_{-1}, B_0 and B_1 are the only annuli in $\{B_n\}_{n\in\mathbb{Z}}$ that are not compactly contained in the corresponding annulus A_n because we have $\mu(R_+)/\varepsilon = M(R_+)$. In our construction B_0 is exceptional because $f(B_0)$ does not necessarily cover any B_n (not even itself) whereas all the others at least cover themselves and the following one.

Theorem 2.6 describes what types of orbits can be found using the covering properties of the annuli B_n that we just constructed.

Proof of Theorem 2.6. By Lemma 2.17, if $f(B_{s_n}) \supseteq B_{s_{n+1}}$, for all $n > 0$, then there is a point $z_0 \in B_{s_0} \subseteq A_{s_0}$ such that $f^n(z_0) \in B_{s_n} \subseteq A_{s_n}$, for all $n > 0$.

We will now describe sequences that produce the various types of annular itinerary listed in Theorem 2.6, with diagrams to illustrate each of these.

- The partition $\{A_n\}_{n\in\mathbb{Z}}$ is convenient for describing fast escaping points that escape to one of the essential singularities. These correspond to annular itineraries where $s_{n+1} = s_n + 1$, if $e = \overline{\infty}$, or $s_{n+1} = s_n - 1$, if $e = \overline{0}$, for n arbitrarily large.

$$\begin{tikzcd} \cdots & A_{-2} \arrow[l, "f"'] & A_{-1} \arrow[l, "f"'] & A_0 & A_1 \arrow[r, "f"] & A_2 \arrow[r, "f"] & \cdots \end{tikzcd}$$

- In order to construct a point with a periodic annular itinerary $s = \overline{s_1 s_2 \cdots s_n}$ we require that $s_{i+1} \in \{k_i, \ldots, s_i + 1\} \setminus \{0\}$, if $s_{i+1} > 0$, or $s_{i+1} \in \{s_i - 1, \ldots, k_i\} \setminus \{0\}$, if $s_{i+1} < 0$, for all $1 \leqslant i \leqslant n$.

$$\begin{tikzcd} A_{n_2-2} \arrow[rrrr, bend left, "f"] & A_{n_2-1} \arrow[l, "f"] & A_{n_2} \arrow[l, "f"] & A_0 & A_{n_1} \arrow[r, "f"] & A_{n_1+1} \arrow[r, "f"] & A_{n_1+2} \arrow[llll, bend left, "f"] \end{tikzcd}$$

- We can construct bounded itineraries whose entries are all s_n or s_{n+1} that are not periodic as follows. We can always choose to stay in the same annulus (every B_n covers itself) or go one level up or down.

$$\begin{tikzcd} A_n \arrow[loop left, "f"] \arrow[r, bend left, "f"] & A_{n+1} \arrow[l, bend left, "f"] \arrow[loop right, "f"] \end{tikzcd}$$

The claim that there are uncountably many such itineraries follows from the fact that at each step we always have two choices. Thus there is a bijection between this set and $2^{\mathbb{N}_0} \cong [0, 1]$.

- Unbounded non-escaping itineraries are those for which there is a sequence (n_k) such that $|s_{n_k}| \to \infty$ as $k \to \infty$ and another sequence (m_k) such that for all $k \in \mathbb{N}_0$, $|m_k| < R$ for some $R > 0$.

$$\begin{tikzcd}
A_n \arrow[r, bend left=60, "f"] \arrow[r, bend left=40, "f"] \arrow[r, bend left=15, "f"] & A_{n+1} \arrow[l, bend left=30, "f"] \arrow[r, bend left=60, "f"] \arrow[r, bend left=40, "f"] & A_{n+2} \arrow[ll, bend left=30, "f"] \arrow[r, bend left=60, "f"] & A_{n+3} \arrow[lll, bend left=30, "f"] & \cdots
\end{tikzcd}$$

We are able to construct uncountably many such itineraries because we can always map to the next annulus or map back to either B_1 or B_{-1}.

- Let $e \in \{0, \infty\}^{\mathbb{N}_0}$ and (r_n) be a sequence of positive real numbers such that

$$r_n > 1 \Leftrightarrow e_n = \infty \quad \text{for all } n \in \mathbb{N}_0,$$

and

$$|\log r_n| \to +\infty \quad \text{as } n \to \infty.$$

Then we can construct a point $z_0 \in I_e(f)$ such that $|f^n(z_0)| < r_n$, if $e_n = \infty$, or $|f^n(z_0)| > r_n$, if $e_n = 0$, for all $n \in \mathbb{N}_0$ sufficiently large. To do so, note that each B_n covers itself, so we can choose to stay in B_n, $n > 0$, for as many iterates as we need so that $M\big(\mu^n(R_+)\big) < r_n$ and then we can choose to *jump* to B_{n+1}.

$$\begin{tikzcd}
A_n \arrow[loop above, "f"] \arrow[r, "f"] & A_{n+1} \arrow[loop above, "f"] \arrow[r, "f"] & A_{n+2} \arrow[loop above, "f"] \arrow[r, "f"] & A_{n+3} \arrow[loop above, "f"] \arrow[r, "f"] & \cdots
\end{tikzcd}$$

This concludes the proof of Theorem 2.6. ∎

2.5 eremenko's properties

In the previous section, we proved the existence of points that escape as fast as possible to infinity and to zero but we are interested in having general fast escaping points in the sense of Definition 2.3, which includes points that *jump* infinitely many times between a neighbourhood of infinity and a neighbourhood of zero. For this, we will modify the construction used to prove Theorem 2.18 in order to mix the iterates of $M(r)$ and $m(r)$.

Proof of Theorem 2.4. Let $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ be an essential itinerary and let $R_0 > 0$ be chosen sufficiently large or small according to whether $e_0 = \infty$ or $e_0 = 0$. Consider the sequence given by, for $n > 0$,

- $R_n = M(R_{n-1})$, if $e_n = \infty$,
- $R_n = m(R_{n-1})$, if $e_n = 0$.

We will show that there is a point z such that, for all $n \in \mathbb{N}_0$,

- $|f^n(z)| \geqslant R_n$, if $e_n = \infty$,
- $|f^n(z)| \leqslant R_n$, if $e_n = 0$.

Hence $z \in A_e(f)$ (note that here $\ell = k = 0$). For this we will also require an auxiliary sequence $(\widetilde{R_n})$ that combines the iterates of $\mu(r) = \varepsilon M(r)$ and $\nu(r) = m(r)/\varepsilon$, where $0 < \varepsilon < 1$, according to the essential itinerary $e \in \{0, \infty\}^{\mathbb{N}}$. Let $\widetilde{R_0} = \mu(R_0) > R_0$, if $e_0 = \infty$, and $\widetilde{R_0} = \nu(R_0) < R_0$, if $e_0 = 0$, so that the sequence $(\widetilde{R_n})$ has a head start on (R_n). For $n > 0$, let

- $\widetilde{R_n} = \mu(\widetilde{R_{n-1}})$, if $e_n = \infty$,
- $\widetilde{R_n} = \nu(\widetilde{R_{n-1}})$, if $e_n = 0$.

Lemma 2.14 guarantees that if $R_0 > R(f)$ or $R_0 < 1/R(f)$, then the sequence (R_n) accumulates to $\{0, \infty\}$ according to the essential itinerary e. For $0 < \varepsilon < 1$, let $R(f, \varepsilon) \geqslant R(f)$ be such that, for instance, $\varepsilon r^2 > r^{3/2}$ for all $r > R(f, \varepsilon)$ and $r^2/\varepsilon < r^{3/2}$ for all $0 < r < 1/R(f, \varepsilon)$. Then the sequence $(\widetilde{R_n})$ also escapes provided that $R_0 > R(f, \varepsilon)$ or $0 < R_0 < 1/R(f, \varepsilon)$.

Proceeding in the same way as in the proof of Theorem 2.18, we define a sequence of closed annuli (B_n) such that $f(B_n) \supseteq B_{n+1}$. First, for $n > 0$, we put

$$B_n := \overline{A}\big(\varepsilon \widetilde{R_n}, \widetilde{R_n}/\varepsilon\big) = \begin{cases} \overline{A}\big(\varepsilon^2 M(\widetilde{R_{n-1}}), M(\widetilde{R_{n-1}})\big), & \text{if } e_n = \infty, \\ \overline{A}\big(m(\widetilde{R_{n-1}}), m(\widetilde{R_{n-1}})/\varepsilon^2\big), & \text{if } e_n = 0, \end{cases} \tag{2.2}$$

where $0 < \varepsilon < 1$ has been chosen suitably small. Next we argue as in Lemma 2.12 to combine the iterates of $M(r)$ and $m(r)$, assuming that the value of r is large enough or small enough, to obtain, for $n > 0$,

$$\begin{aligned} R_n &< \varepsilon^2 M(\widetilde{R_{n-1}}), &\quad \text{if } e_n = \infty, \\ m(\widetilde{R_{n-1}})/\varepsilon^2 &< R_n, &\quad \text{if } e_n = 0. \end{aligned} \tag{2.3}$$

We prove (2.3) by induction. The base case $n = 1$ holds provided that the number R_0 (and hence R_1) is large enough, or small enough:

$$\begin{aligned} R_1 < \varepsilon^2 M(\widetilde{R_0}) &= \varepsilon^2 M(\varepsilon R_1), &\quad \text{if } e_1 = \infty \text{ and } e_0 = \infty, \\ R_1 < \varepsilon^2 M(\widetilde{R_0}) &= \varepsilon^2 M(R_1/\varepsilon), &\quad \text{if } e_1 = \infty \text{ and } e_0 = 0, \\ m(\varepsilon R_1)/\varepsilon^2 &= m(\widetilde{R_0})/\varepsilon^2 < R_1, &\quad \text{if } e_1 = 0 \text{ and } e_0 = \infty, \\ m(R_1/\varepsilon)/\varepsilon^2 &= m(\widetilde{R_0})/\varepsilon^2 < R_1, &\quad \text{if } e_1 = 0 \text{ and } e_0 = 0. \end{aligned}$$

Let $e_n = \infty$ and $e_{n+1} = 0$, and suppose $R_n < \varepsilon^2 M(\widetilde{R_{n-1}})$. Then,

$$m(\widetilde{R_n}) = m\big(\varepsilon M(\widetilde{R_{n-1}})\big) < \varepsilon^2 m\big(\varepsilon^2 M(\widetilde{R_{n-1}})\big) < \varepsilon^2 m(R_n) = \varepsilon^2 R_{n+1},$$

as required. Note that the first inequality here is due to the fact that $m(r) < \varepsilon^2 m(\varepsilon r)$ for $0 < \varepsilon < 1$ and $r > 0$ sufficiently large (see Lemma 2.10(iv)), while in the second inequality we use the induction hypothesis. The other three possible combinations of e_n and e_{n+1} follow similarly.

Thus, we have, for $n > 0$, by (2.2) and (2.3),

$$\begin{aligned} B_n \subseteq A\big(R_n,\, M(R_n)\big) &\subseteq \mathbb{C} \setminus D(0, R_n), &\quad \text{if } e_n = \infty, \\ B_n \subseteq A\big(m(R_n),\, R_n\big) &\subseteq D(0, R_n), &\quad \text{if } e_n = 0. \end{aligned}$$

Thus, taking $z_1, z_2 \in B_n$ such that $|z_1| = |z_2| = \widetilde{R_n}$ and

$$|f(z_1)| = m(\widetilde{R_n}) \quad \text{and} \quad |f(z_2)| = M(\widetilde{R_n}),$$

and applying Lemma 2.19 we deduce that $f(B_n)$ always covers B_{n+1} and therefore, by Lemma 2.17, $A_e(f) \neq \emptyset$.

Finally, Baker showed that transcendental self-maps of $\mathbb{C}^*$ can only have one doubly connected Fatou component, which separates zero from infinity [Bak87, Theorem 1]. Thus, $B_n \cap J(f) \neq \emptyset$ for all n large enough and, since $J(f)$ is a backward invariant closed set, we have $A_e(f) \cap J(f) \neq \emptyset$. ■

Next we prove Theorems 2.7 and 2.8 which correspond to properties (I2) and (I3) proved by Eremenko in [Ere89] for transcendental entire functions. To that end, in Theorem 2.9 we show that, for a transcendental self-map of $\mathbb{C}^*$, the components of the fast escaping set are unbounded in $\mathbb{C}^*$. Before this, we prove Theorem 2.21 which concerns fast escaping Fatou components and is of independent interest as well as being a key ingredient of the proof of Theorem 2.7.

Theorem 2.21. *Let f be a transcendental self-map of $\mathbb{C}^*$. If U is a Fatou component of f such that $U \cap A_e^{-\ell,k}(f,R) \neq \emptyset$ for some $R > 0$, $\ell \geqslant 1$, $k \in \mathbb{N}_0$ and $e \in \{0,\infty\}^{\mathbb{N}_0}$, then $\overline{U} \subseteq A_e^{-\ell,k}(f,R)$.*

Note that if $\ell < 1$, then $A_e^{-\ell,k}(f) \subseteq A_e^{-1,k-\ell+1}(f)$, by Lemma 2.15, so we can apply Theorem 2.21 to the set $A_e^{-1,k-\ell+1}(f)$. To prove Theorem 2.21 we will use the following version of a distortion lemma of Baker [Bak88] (see also [Ber93, Lemma 7] and [RS00, Theorem 3]) adapted to $\mathbb{C}^*$.

Lemma 2.22. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let U be a Fatou component that is in $I(f)$. Let K be a compact subset of U. Then there exist constants $C > 1$ and $n_0 \in \mathbb{N}_0$ such that*

$$|f^n(z_1)| \leqslant C|f^n(z_2)|$$

for all $z_1, z_2 \in K$ and $n \geqslant n_0$.

Proof. Let $U_n := f^n(U)$ for all $n \in \mathbb{N}_0$, which are not necessarily distinct. Recall that $[z,w]_\Omega$ is the hyperbolic distance between two points z and w in a hyperbolic domain Ω; we use $\rho_\Omega(z)$ to denote the hyperbolic density in Ω. By Theorem 2.4 and Theorem 2.9, we can choose two connected subsets X_0 and X_∞ of $A(f) \cap J(f)$ whose closure in $\hat{\mathbb{C}}$ contains, respectively, zero and infinity, and such that $G := \mathbb{C}^* \setminus (X_0 \cup X_\infty)$ is simply connected. For all $z_1, z_2 \in K$,

$$[z_1,z_2]_{U_0} \geqslant [f^n(z_1), f^n(z_2)]_{U_n} \geqslant [f^n(z_1), f^n(z_2)]_G = \int_{\Gamma_n} \rho_G(z)|dz|,$$

where Γ_n is a hyperbolic geodesic in G joining $f^n(z_1)$ to $f^n(z_2)$. Since the set G is simply connected, there exists a constant $c > 0$ such that, for sufficiently large $R > 0$,

$$\rho_G(z) \geqslant \frac{c}{|z|} \quad \text{for } |z| > R \text{ and } |z| < 1/R.$$

This follows from [CG93, Theorem 4.3 in Chapter I] for z near zero. Observe that, for z near infinity, taking $g(z) := 1/z$, we obtain

$$\rho_G(z) = \rho_{g(G)}(g(z))|g'(z)| = \frac{\rho_{g(G)}(1/z)}{|z|^2} \geqslant \frac{c|z|}{|z|^2} = \frac{c}{|z|} \quad \text{for } |z| > R.$$

Let $n_0 \in \mathbb{N}_0$ be such that both points $f^n(z_1), f^n(z_2)$ are contained in $D(0, 1/R)$ or $\mathbb{C} \setminus \overline{D(0,R)}$ for all $n \geqslant n_0'$. Thus, if $|f^n(z_2)| \leqslant |f^n(z_1)|$, we have

$$[f^n(z_1), f^n(z_2)]_G \geqslant \int_{|f^n(z_2)|}^{|f^n(z_1)|} \frac{c}{r} dr = c \ln \frac{|f^n(z_1)|}{|f^n(z_2)|}.$$

Hence,

$$\frac{|f^n(z_1)|}{|f^n(z_2)|} \leqslant \exp\left(\frac{1}{c}[z_1, z_2]_{U_0}\right) =: C$$

and $|f^n(z_1)| \leqslant C|f^n(z_2)|$ for $n \geqslant n_0$ as required. ∎

We now prove Theorem 2.21 concerning fast escaping Fatou components of transcendental self-maps of $\mathbb{C}^*$.

Proof of Theorem 2.21. Suppose first that U is simply connected. We can assume, without loss of generality, that $k = 0$, otherwise take a shift of e. We can also assume that there is a sequence (n_j) such that $e_{n_j} = \infty$, as the proof is similar in the other case.

Let $R > R(f)$ and consider the sequence of real numbers (R_n) that starts with $R_0 := R$, if $e_0 = \infty$, or $R_0 := 1/R$, if $e_0 = 0$, and is defined iteratively by $R_n = \widetilde{M}(R_{n-1})$ as in Definition 2.3, where $\widetilde{M}(r)$ is $M(r)$ or $m(r)$ according to e. If $z_0 \in U \cap A_e^{-\ell,0}(f, R)$, then

$$|f^{n_j+\ell}(z_0)| \geqslant \widetilde{M}^{n_j}(R_0) = R_{n_j} \quad \text{for } j \in \mathbb{N}_0. \tag{2.4}$$

Suppose now that there is $z_1 \in U \setminus A_e^{-\ell,0}(f, R)$. By normality, the essential itineraries of z_0 and z_1 need to coincide eventually, that is, there is $L \in \mathbb{N}_0$ such that $f^{L+\ell}(z_0)$ and $f^{L+\ell}(z_1)$ have the same essential itinerary, and L is the smallest value with this property. Now, either there is $L' \geqslant L$ such that

$$|f^{L'+\ell}(z_1)| < R_{L'}, \text{ if } e_{L'} = \infty, \quad \text{or} \quad |f^{L'+\ell}(z_1)| > R_{L'}, \text{ if } e_{L'} = 0,$$

or, for all $n \geqslant L$,

$$|f^{n+\ell}(z_1)| \geqslant R_n, \text{ if } e_n = \infty, \quad \text{or} \quad |f^{n+\ell}(z_1)| \leqslant R_n, \text{ if } e_n = 0.$$

Note that, since $z_1 \notin A_e^{-\ell,0}(f, R)$, if $L = 0$, then the first case applies. If $L > 0$ and we are in the second case, then, by continuity, there exists $z_2 \in f^{L+\ell-1}(U)$ with the same essential itinerary as $f^{L+\ell-1}(z_0)$ and such that

$$|z_2| < R_{L-1}, \text{ if } e_{L-1} = \infty, \quad \text{or} \quad |z_2| > R_{L-1}, \text{ if } e_{L-1} = 0.$$

Hence, we can suppose that the first case applies and, if necessary, continue the argument with the iterates of the point z_2 instead of those of $f^{L+\ell-1}(z_1)$. Thus, by Lemma 2.14(ii), there is $N = n_m \geqslant L$ for some $m \in \mathbb{N}_0$ and $c > 1$ such that

$$R(f) < |f^{N+\ell}(z_1)| = \widetilde{M}^N(R_0)^{1/c} = R_N^{1/c} =: K$$

and hence, by the definition of $\widetilde{M}(r)$,

$$|f^{n_j+\ell}(z_1)| = \left|f^{n_j-N}\left(f^{N+\ell}(z_1)\right)\right| \leqslant \widetilde{M}^{n_j-N}(R_N^{1/c}) \quad \text{for } n_j > N. \tag{2.5}$$

We can suppose that K is larger than the constant $R = R^{\infty}(f)$ from Lemma 2.10(iii). Then, combining equations (2.4) and (2.5), we obtain

$$\frac{|f^{n_j+\ell}(z_0)|}{|f^{n_j+\ell}(z_1)|} \geqslant \frac{\widetilde{M}^{n_j}(R_0)}{\widetilde{M}^{n_j-N}(R_N^{1/c})} = \frac{\widetilde{M}^{n_j-N}\left(\widetilde{M}^N(R_0)\right)}{\widetilde{M}^{n_j-N}(K)} = \frac{\widetilde{M}^{n_j-N}(K^c)}{\widetilde{M}^{n_j-N}(K)}$$

for all $n_j > N$. This contradicts Lemma 2.22 because, applying Lemma 2.10(iii) repeatedly,

$$\frac{\widetilde{M}^{n_j-N}(K^c)}{\widetilde{M}^{n_j-N}(K)} \geqslant \frac{\left(\widetilde{M}^{n_j-N}(K)\right)^c}{\widetilde{M}^{n_j-N}(K)} = \left(\widetilde{M}^{n_j-N}(K)\right)^{c-1} \to +\infty \text{ as } j \to \infty.$$

Therefore $U \subseteq A_e^{-\ell,0}(f, R)$ and, since $A_e^{-\ell,0}(f, R)$ is closed, we have that $\overline{U} \subseteq A_e^{-\ell,0}(f, R)$.

Finally, if U is not simply connected, then $f(U)$ is simply connected since f can only have one multiply connected Fatou component [Bak87, Theorem 1]. As $f\big(A_e^{-\ell,k}(f,R)\big) \subseteq A_e^{-\ell+1,k}(f,R)$, in this case $f(U) \cap A_e^{-\ell+1,k}(f,R) \neq \emptyset$ and using the same argument as above we conclude that $\overline{f(U)} \subseteq A_e^{-\ell+1,k}(f,R)$. Since we supposed that $\ell \geqslant 1$, in this case we also have

$$\overline{U} \subseteq f^{-1}\big(A_e^{-\ell+1,k}(f,R)\big) = A_e^{-\ell,k}(f,R)$$

as required. ∎

Now we can prove Theorem 2.7, which says that, for each sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, we have that $J(f) = \partial A_e(f) = \partial I_e(f)$ and also $J(f) = \partial A(f) = \partial I(f)$.

Proof of Theorem 2.7. Take $z \in J(f)$, and let V be a neighbourhood of z. Consider a point $z_1 \in A_e(f) \subseteq I(f)$. Since the family of iterates of f is not normal in V, by Montel's theorem we can find a preimage z^* of z_1 in V; that is, $f^k(z^*) = z_1$ for some $k \geqslant 1$. Since $A_e(f)$ and $I(f)$ are completely invariant, $z^* \in A_e(f) \subseteq I(f)$. Thus $J(f) \subseteq \overline{A_e(f)}$. But $\operatorname{int} A_e(f) \subseteq \operatorname{int} I(f) \subseteq F(f)$ because periodic points are dense in $J(f)$. So $J(f) \subseteq \partial A_e(f)$.

The opposite inclusion follows from Theorem 2.21. If there exists a point $z \in \partial A_e(f) \cap F(f)$, then there would be points arbitrarily close to z in $A_e(f)$ but since $F(f)$ is open the whole Fatou component would be in $A_e(f)$. Hence $\partial A_e(f) \subseteq J(f)$.

The facts that $J(f) = \partial I_e(f)$ for each essential itinerary $e \in \{0,\infty\}^{\mathbb{N}_0}$ and $J(f) = \partial A(f) = \partial I(f)$ are proved similarly. ∎

Observe that $\{A_e(f)\}$ and $\{I_e(f)\}$ contain uncountable collections of disjoint sets all sharing the same boundary, which is precisely the Julia set $J(f)$. Baker, Domínguez and Herring [BDH01] had shown previously that if f is a meromorphic function with a certain set of essential singularities E then the set of points escaping to one particular $e \in E$, namely $I(f,e)$, satisfies that $\partial I(f,e) = J(f)$. This implies that in our setting $\partial I_0(f) = \partial I_\infty(f) = J(f)$ which was also shown by Fang [Fan98]. Our result shows that this property holds for $I_e(f)$ for any essential itinerary $e \in \{0,\infty\}^{\mathbb{N}_0}$.

Next we prove Theorem 2.8. Recall that a set X is unbounded in $\mathbb{C}^*$ if $\widehat{X} \cap \{0,\infty\} \neq \emptyset$.

Proof of Theorem 2.8. Suppose to the contrary that X is a component of $\overline{I_e(f)}$ that is bounded away from zero and infinity. Then there is a topological annulus A in the complement of $I_e(f)$ separating X from both zero and infinity. Since the points in A have orbits that miss the infinite set $I_e(f)$, $A \subseteq F(f)$ by Montel's theorem. Let K be the component of $\mathbb{C}^* \setminus A$ containing X. By Theorem 2.7, we have $K \cap J(f) \neq \emptyset$ and hence A must be contained in a multiply connected component of $F(f)$. But Baker and Domínguez [BD98] showed that such components must be doubly connected and separate zero from infinity which is a contradiction to the fact that A is doubly connected and separates a component of $J(f)$ from both zero and infinity.

The last claim in the statement of the theorem follows from the fact that every connected component of $\overline{I(f)}$ contains at least one component of $\overline{I_e(f)}$ for some $e \in \{0, \infty\}^{\mathbb{N}_0}$, and hence it must be unbounded as well. ∎

Before proving Theorem 2.9 we need the following lemma concerning preimages of unbounded closed sets under transcendental self-maps of $\mathbb{C}^*$.

Lemma 2.23. *Let f be a transcendental self-map of $\mathbb{C}^*$, and let $X \subseteq \mathbb{C}^*$ be an unbounded continuum. Then all the components of $f^{-1}(X)$ are unbounded.*

Proof. Let W be a connected component of $f^{-1}(X)$. Since f is continuous and X is closed, $f^{-1}(X)$ is also closed. Assume, to the contrary, that W is bounded. Then, by [New61, p. 143], there exists an annulus $A \supseteq W$, whose boundary consists of Jordan curves in $\mathbb{C}^* \setminus f^{-1}(X)$. Since f is an open mapping, $f(A)$ is a connected open set in $\mathbb{C}^*$ and $\partial f(A) \subseteq f(\partial A)$, which does not meet X. Thus $X \subseteq f(A)$, which contradicts the fact that X is unbounded. ∎

Finally we prove Theorem 2.9 which says that the connected components of $A(f)$ are all unbounded. Note that, in particular, this implies that $I(f)$ has at least one component which has zero in its closure and one component (possibly the same component) which has infinity in its closure.

Proof of Theorem 2.9. Let $z_0 \in A_e(f)$, then $z_0 \in A_e^{-\ell,k}(f, R)$ for some $\ell \in \mathbb{Z}$, $k \in \mathbb{N}_0$ and $R > 0$. By Lemma 2.15, it is enough to consider the case $\ell \in \mathbb{N}$, and, by Lemma 2.23, if $f^{\ell}(z_0)$ lies in an unbounded component X of $A_e(f)$, then $f^{-\ell}(X)$ is also unbounded, and thus we can assume that $\ell = 0$. Furthermore, since $A_e^{-\ell,k}(f,R) = A_{\sigma^k(e)}^{-\ell,0}(f,R)$, we can also suppose that $k = 0$. Fix $n \in \mathbb{N}_0$ and suppose that $e_n = \infty$, that is,

$$|f^n(z_0)| > R_n = M(R_{n-1}).$$

Consider the finite sequence of closed sets

$$X_{n,j} := f^{-j}\big(\mathbb{C} \setminus D(0,R_n)\big), \quad j = 1,\dots,n,$$

which, by Lemma 2.23, are unbounded in $\mathbb{C}^*$. One of the connected components of $X_{n,j}$ must contain the point $f^{n-j}(z_0)$; we denote this component by $L_{n,j}$.

Now there are two cases to consider: either

(i) $e_{n-1} = \infty$ and $|f^{n-1}(z_0)| > M(R_{n-2}) = R_{n-1}$, or

(ii) $e_{n-1} = 0$ and $|f^{n-1}(z_0)| < m(R_{n-2}) = R_{n-1}$.

In case (i), $L_{n,1}$ cannot contain points of modulus less than R_{n-1}. Otherwise if w is such that $|w| < R_{n-1}$ and $f(w) \in L_{n,0} = \mathbb{C} \setminus D(0,R_n)$ then we would get a contradiction with the fact that $R_n = M(R_{n-1})$ but $|f(w)| > R_n$. Similarly, in case (ii), if $|f^{n-1}(z_0)| < R_{n-1}$ we cannot have points in $L_{n,1}$ that have modulus larger than R_{n-1}.

Now, iterating this procedure, for every $n \in \mathbb{N}_0$, we deduce that $L_n = L_{n,n}$ is a closed connected set which is contained in $\mathbb{C} \setminus D(0,R_0)$, if $e_0 = \infty$, or in $D(0,R_0)$, if $e_0 = 0$. Observe that

$$L_{n+1} \subseteq L_n.$$

Otherwise there would exist $w \in L_{n+1}$ such that $w \notin L_n$ which means that

$$\begin{aligned} |f^{n+1}(w)| &> R_{n+1}, \quad \text{if } e_n = \infty, \\ |f^{n+1}(w)| &< R_{n+1}, \quad \text{if } e_n = 0, \end{aligned}$$

but

$$\begin{aligned} |f^{n}(w)| &< R_{n}, \quad \text{if } e_n = \infty, \\ |f^{n}(w)| &> R_{n}, \quad \text{if } e_n = 0, \end{aligned}$$

which is a contradiction. Therefore $(L_n \cup \{e_0\})$ is a nested sequence of continua all containing z_0 and e_0, and hence

$$K = \bigcap_{n \in \mathbb{N}} (L_n \cup \{e_0\})$$

is also a continuum in $\hat{\mathbb{C}}$ which contains z_0 and e_0. Let Γ be the connected component of $K \setminus \{e_0\}$ that contains z_0. Then Γ is closed and unbounded. Here we are using the following result from continuum theory: if E_0 is a continuum in $\hat{\mathbb{C}}$, E_1 is a closed subset of E_0 and C is a component of $E_0 \setminus E_1$, then $\overline{C}$ meets E_1 [New61, p. 84]. Since $\Gamma \subseteq A_e(f)$, the theorem is proved. ∎

3 DYNAMIC RAYS OF BOUNDED-TYPE FUNCTIONS

In this chapter we study the escaping set of functions in the class $\mathcal{B}^*$, that is, transcendental self-maps of $\mathbb{C}^*$ for which the set of singular values is bounded in $\mathbb{C}^*$. For functions in the class $\mathcal{B}^*$, escaping points lie in their Julia set. We prove that if f is a composition of finite order transcendental self-maps of $\mathbb{C}^*$ (and hence, in the class $\mathcal{B}^*$), then every escaping point of f can be connected to one of the essential singularities by a curve of points that escape uniformly. Moreover, for every sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, we show that the set $I_e(f)$ contains an absorbing Cantor bouquet. We also show the existence of periodic dynamic rays, which must land.

3.1 INTRODUCTION AND MAIN RESULTS

In the punctured plane, the analogue of the Eremenko-Lyubich class $\mathcal{B}$ is the class

$$\mathcal{B}^* := \{f \text{ transc. self-map of } \mathbb{C}^* \;:\; \operatorname{sing}(f^{-1}) \text{ is bounded away from } 0, \infty\}$$

which consists of *bounded-type* transcendental self-maps of $\mathbb{C}^*$. We prove the following result for functions in the class $\mathcal{B}^*$.

Theorem 3.1. *Let* $f \in \mathcal{B}^*$. *Then* $I(f) \subseteq J(f)$.

Recall that Kotus[Kot90] showed that transcendental self-maps of $\mathbb{C}^*$ can have Baker domains and wandering domains; we will construct more examples of functions with escaping Fatou components in the next chapter. It remains an open question whether functions in the class $\mathcal{B}^*$ can have wandering domains outside the escaping set, as is the case for entire functions in the class $\mathcal{B}$ [Bis15, Theorem 17.1].

It is a natural question to ask about the relationship between entire functions in the class $\mathcal{B}$ and transcendental self-maps of $\mathbb{C}^*$ in the class $\mathcal{B}^*$. Keen [Kee88] showed that if P and Q are polynomials and $n \in \mathbb{Z}$, then the function $f(z) = z^n \exp\big(P(z) + Q(1/z)\big)$ has a finite number of singular values and hence belongs to the class $\mathcal{B}^*$. The next theorem extends this results to all functions in the class $\mathcal{B}$ when $n = 0$.

Theorem 3.2. *Let* g *and* h *be entire functions in the class* $\mathcal{B}$. *Then the function* $f(z) = \exp\big(g(z) + h(1/z)\big)$ *is in the class* $\mathcal{B}^*$.

In the previous chapter we discussed the properties of the maximum and minimum modulus functions; for $r > 0$, we define

$$M(r, f) := \max_{|z|=r} |f(z)| \quad \text{and} \quad m(r, f) := \min_{|z|=r} |f(z)|.$$

In contrast to the situation for entire functions, there is a strong relation between the bounded-type condition for holomorphic self-maps of $\mathbb{C}^*$ and their order of growth. To be more precise, recall that the *order* and *lower order* of an entire function f can be defined, respectively, as

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log\log M(r, f)}{\log r} \quad \text{and} \quad \lambda(f) := \liminf_{r\to+\infty} \frac{\log\log M(r, f)}{\log r}.$$

If f is a transcendental self-map of $\mathbb{C}^*$, then we also need to take into account the essential singularity at zero. Hence the *order* of growth is given by two quantities:

$$\rho_\infty(f) := \limsup_{r\to+\infty} \frac{\log\log M(r, f)}{\log r}, \quad \rho_0(f) := \limsup_{r\to 0} \frac{\log\log 1/m(r, f)}{\log 1/r}.$$

We say that f has *finite order* if both $\rho_\infty(f) < +\infty$ and $\rho_0(f) < +\infty$. Likewise, we can define two quantities associated with the lower order of such functions, $\lambda_\infty(f)$ and $\lambda_0(f)$, by replacing lim sup by lim inf in the expression above. An important property of entire functions $f \in \mathcal{B}$ is that $\lambda(f) \geqslant 1/2$ [BE95; Lan95] (see also [RS05a, Lemma 3.5]). The next result shows that, surprisingly, the lower order of a function in $\mathbb{C}^*$ always equals its order. Moreover, if the order is finite, then it is an integer.

Theorem 3.3. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$. *Then* $\lambda_0(f) = \rho_0(f)$ *and* $\lambda_\infty(f) = \rho_\infty(f)$. *If* f *has finite order, then* $f(z) = z^n \exp\big(P(z) + Q(1/z)\big)$ *where* $n \in \mathbb{Z}$ *and* P, Q *are polynomials, and therefore* $\rho_0(f), \rho_\infty(f) \in \mathbb{Z}$ *and* $f \in \mathcal{B}^*$. *In particular,* $\lambda_0(f), \lambda_\infty(f) \geqslant 1$.

Rottenfußer, Rückert, Rempe and Schleicher [RRRS11, Theorem 1.2] proved that the stronger version of Eremenko's conjecture holds for transcendental entire functions of bounded type and finite order or, more generally, a finite composition of such functions: every escaping point can be joined to infinity by a curve of points that escape uniformly; such curves are called *ray tails* and their maximal extensions are called *dynamic rays*. This result was proved independently by Barański [Bar07, Theorem C] for *disjoint-type* functions, that is, transcendental entire functions for which the Fatou set consists of a completely invariant component which is a basin of attraction. In general, it is not known whether such dynamic rays must be smooth.

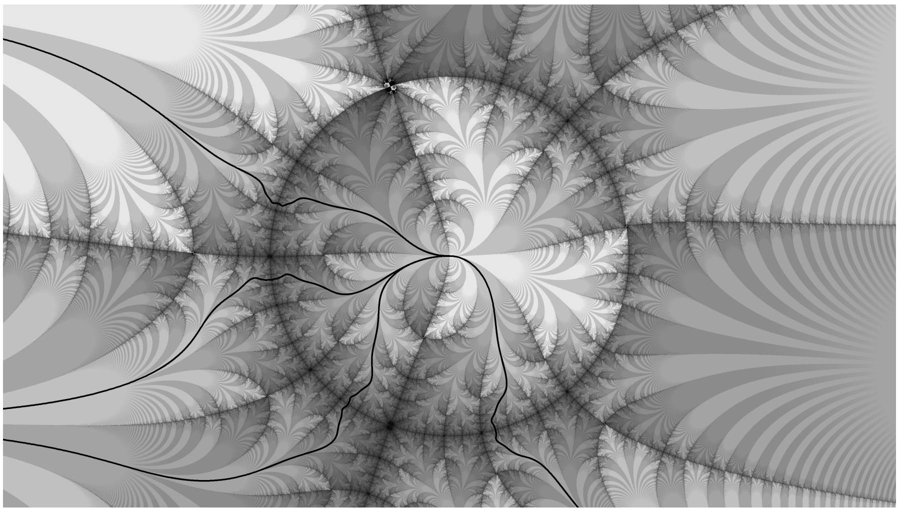


Figure 4: Period 8 cycle of rays landing on a repelling period 4 orbit in the unit circle for the function $f_{\alpha\beta}(z) = ze^{i\alpha}e^{\beta(z-1/z)/2}$ from the Arnol'd standard family, with $\alpha = 0.19725$ and $\beta = 0.48348$.

We prove the existence of dynamic rays for transcendental self-maps of $\mathbb{C}^*$ by adapting the construction of [RRRS11] to our setting. We use the notation $f^n_{|\gamma} \to \{0,\infty\}$ to mean that, under iteration by f, all the points in γ escape to zero, escape to infinity or accumulate at both of them and nowhere else.

Theorem 3.4. *Let f be a transcendental self-map of $\mathbb{C}^*$ of finite order or, more generally, a finite composition of such functions. Then every point $z \in I(f)$ can be connected to either zero or infinity by a curve γ such that $f^n_{|\gamma} \to \{0,\infty\}$ uniformly in the spherical metric as $n \to \infty$.*

Note that in the statement of Theorem 3.4 there is no assumption of bounded type. This is because finite order transcendental self-maps of $\mathbb{C}^*$ are always in the class $\mathcal{B}^*$ (see Lemma 3.35).

Remark 3.5. In the masters thesis [Mar11], we showed that points in the sets $I_0(f)$ and $I_\infty(f)$ can be joined to zero and infinity, respectively, by ray tails. Theorem 3.4 generalises this result to all points in $I(f)$.

Bergweiler [Ber95] proved that if $\tilde{f}$ is a lift of a holomorphic self-map f of $\mathbb{C}^*$, then $J(\tilde{f}) = \exp^{-1} J(f)$. Seeing this result one might think that every result about entire functions could be extended to self-maps of $\mathbb{C}^*$ via their lifts. Unfortunately, this is not possible. In particular, a lift of a map of bounded type is never of bounded type, its singular set is contained in a vertical band and so, we cannot apply directly the results from [RRRS11]. However, Theorem 3.4 allows to construct dynamic rays for certain entire functions that are not in the class $\mathcal{B}$, but project to functions in the class $\mathcal{B}^*$ satisfying the hypothesis of Theorem 3.4.

Corollary 3.6. *Let f be an entire transcendental function of finite order such that there exists $k \in \mathbb{Z}$ so that $f(z+2\pi i) = f(z) + k2\pi i$ for all $z \in \mathbb{C}$, or a finite composition of such functions. Then every point $z \in I(f)$ with $|\operatorname{Re} f^n(z)| \to +\infty$ as $n \to \infty$ can be connected to infinity by a curve of points that escape uniformly.*

The main tool to prove Theorem 3.4 is the use of logarithmic coordinates, introduced by Eremenko and Lyubich [EL92], and the expansivity of logarithmic transforms. The orbit of escaping points eventually enters the tracts (unbounded Jordan domains which are mapped to a neighbourhood of zero or infinity) and remains there. We partition each tract into *fundamental domains* and consider itineraries on them; see Section 3.5 for the precise definitions. Observe that the previous theorem contains no claim of which dynamic rays actually exist. Our next result shows that, under the hypothesis of Theorem 3.4, there is a unique dynamic ray for every sequence of fundamental domains that contains only finitely many symbols. Here $P(f)$ denotes the *postsingular set* of f which is the closure of the union of all the (forward) iterates of $\operatorname{sing}(f^{-1})$. We say that a dynamic ray γ *lands* if $\overline{\gamma} \setminus \gamma$ is a single point.

Theorem 3.7. *Let f be a transcendental self-map of $\mathbb{C}^*$ of finite order or, more generally, a finite composition of such functions, and let (D_n) be an* *admissible sequence of fundamental domains of* f *containing finitely many symbols. Then the function* f *has a unique non-empty dynamic ray* γ *with itinerary* (D_n)*. Furthermore, if* (D_n) *is periodic and the set* $P(f)$ *is bounded in* $\mathbb{C}^*$*, then the dynamic ray* γ *lands.*

Observe that, for example, Theorem 3.7 implies that every fundamental domain contains exactly one invariant dynamic ray. In the previous chapter we studied a partition of $I(f)$ into non-empty sets $I_e(f)$, for $e \in \{0, \infty\}^{\mathbb{N}_0}$ (see Theorem 2.4). Since there are uncountably many disjoint sets $I_e(f)$, it follows from Theorem 3.4 that $I(f)$ contains uncountably many ray tails. However, each of the sets $I_e(f)$ should be regarded as the analogue of the whole of $I(f)$ for a transcendental entire function f. Thus, we follow the methods of Barański, Jarque and Rempe [BJR12] to prove that, in fact, under the hypothesis of Theorem 3.4, each set $I_e(f)$ contains an absorbing *Cantor bouquet* and, in particular, uncountably many ray tails.

Theorem 3.8. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$ *of finite order or, more generally, a finite composition of such functions. For each sequence* $e \in \{0, \infty\}^{\mathbb{N}_0}$*, there exists a Cantor bouquet* $X_e \subseteq I_e(f)$ *and, in particular, the set* $I_e(f)$ *contains uncountably many ray tails.*

Although Theorem 3.4 is stated in terms of functions of finite order, its proof is more general and applies to a class of functions satisfying certain *good geometry properties* (see Definition 3.28). Rempe, Rippon and Stallard showed that, assuming an extra condition (namely, that the tracts have what they call *bounded gulfs*), the ray tails constructed in [RRRS11] consist of fast escaping points [RRS10, Theorem 1.2]. It seems likely that similar conditions would imply that the dynamic rays that we construct here are also fast escaping in the sense described in the previous chapter.

Remark 3.9. Lasse Rempe-Gillen (private communication) pointed out that Theorem 3.4 may also be proved using *random iteration* as described in the last paragraph of [RRRS11, Section 5] by taking, for $R > 0$ sufficiently large,

$$f_1(z) := \begin{cases} f(z), & \text{if } |f(z)| > R, \\ 1/f(z), & \text{if } |f(z)| < 1/R; \end{cases} \quad f_2(z) := \begin{cases} f(1/z), & \text{if } |f(1/z)| > R, \\ 1/f(1/z), & \text{if } |f(1/z)| < 1/R; \end{cases}$$

which both have a logarithmic transform in the class $\mathcal{B}_{\log}$ and then apply the results of [RRRS11] to a non-autonomous sequence of these two functions. However, it seems natural to provide a direct proof.

Structure of the chapter. Roughly speaking, the first half of the chapter is devoted to describing the basic properties of functions in the class $\mathcal{B}^*$ and in the second half we investigate the existence of dynamic rays for these functions. In Section 3.2, we study what is the relation between the classes $\mathcal{B}$ and $\mathcal{B}^*$; the proof of Theorem 3.2 is there. In Section 3.3, we describe the geometry of logarithmic coordinates of functions in the class $\mathcal{B}^*$ and give the proof of Theorem 3.1. Finite order functions are introduced in Section 3.4, where we prove Theorem 3.3, and are shown to be examples of functions with good geometry. In Section 3.5, we introduce external addresses and describe their relation with essential itineraries. In contrast to what happens in the entire case, in our setting the Bernoulli shift map is a subshift of finite type, where only some sequences are admissible. In Section 3.6, we show that if an external address $\underline{s}$ is periodic, then the set $J_{\underline{s}}(F)$ consisting of all points whose external address is $\underline{s}$ contains an unbounded continuum of fast escaping points - this is used later to prove Theorem 3.7 in Section 3.9. Dynamic rays are introduced in Section 3.7. Finally the proofs of Theorem 3.4 and Theorem 3.8 are sketched in Section 3.8 and Section 3.9, respectively, focusing on the differences with the proofs of [RRRS11, Theorem 1.2] and [BJR12, Theorem 1.6], which concern entire functions.

3.2 FUNCTIONS IN THE CLASS $\mathcal{B}^*$

Let f be a transcendental entire function or a transcendental self-map of $\mathbb{C}^*$. Recall that $v \in \hat{\mathbb{C}}$ is a *critical value* of f if $v = f(c)$ with $f'(c) = 0$, and $a \in \hat{\mathbb{C}}$ is an *asymptotic value* of f if there is a continuous injective curve $\gamma : (0, +\infty) \to \hat{\mathbb{C}}$ (the *asymptotic path*) such that $\gamma(t) \to \alpha$ as $t \to +\infty$, where α is an essential singularity of f, and $f\big(\gamma(t)\big) \to a$. Let $CP(f)$ denote the set of critical point of f. The set of singularities of the inverse function, $\operatorname{sing}(f^{-1})$, consists of the critical values of f, $\mathrm{CV}(f) := f(\mathrm{CP}(f))$, and the finite asymptotic values of f, $\mathrm{AV}(f)$, that is

$$\operatorname{sing}(f^{-1}) = \mathrm{CV}(f) \cup \mathrm{AV}(f).$$

In $\mathbb{C}^*$, by finite asymptotic value we mean that $a \notin \{0, \infty\}$. For transcendental self-maps of $\mathbb{C}^*$, we can decompose $\mathrm{AV}(f)$ as

$$\mathrm{AV}(f) = \mathrm{AV}_0(f) \cup \mathrm{AV}_\infty(f),$$

depending on whether $a \in \mathrm{AV}(f)$ has an asymptotic path γ to zero or to infinity. Note that the set $\mathrm{AV}_0(f) \cap \mathrm{AV}_\infty(f)$ may be non-empty. Finally, we define the *singular set* of f, $S(f)$, and the *postsingular set* of f, $P(f)$, as

$$S(f) := \overline{\operatorname{sing}(f^{-1})}, \quad P(f) := \overline{\bigcup_{n \in \mathbb{N}} f^n\big(\operatorname{sing}(f^{-1})\big)}.$$

We say that f has *bounded type* if $S(f)$ is bounded. Similarly, we say that f has *finite type* if $S(f)$ is finite.

The next result relates the singular set and the postsingular set of a transcendental self-map f of $\mathbb{C}^*$ with the corresponding sets of a lift $\tilde{f}$ of f. Recall that a lift of f is a transcendental entire function $\tilde{f}$ so that $\exp \circ \tilde{f} = f \circ \exp$. The proof is straightforward and we omit it.

Lemma 3.10. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of f. Then $S(\tilde{f}) = \exp^{-1}\big(S(f)\big)$ and $P(\tilde{f}) \subseteq \exp^{-1}\big(P(f)\big)$.*

Recall that if f is a holomorphic self-map of $\mathbb{C}^*$, we define $\operatorname{ind}(f)$ to be the index of $f(\gamma)$ with respect to the origin, where γ is any positively oriented simple closed curve around the origin. Observe that, in the hypothesis of the previous lemma, if $|\operatorname{ind}(f)| = 1$, then $P(\tilde{f}) = \exp^{-1}\big(P(f)\big)$.

The following lemma is a basic property about the singular values of the composition of two functions.

Lemma 3.11. *Let f and g be meromorphic functions in $\mathbb{C}$. Then we have that $\mathrm{CP}(g \circ f) = \mathrm{CP}(f) \cup f^{-1}\big(\mathrm{CP}(g)\big)$, $\mathrm{CV}(g \circ f) \subseteq g\big(\mathrm{CV}(f)\big) \cup \mathrm{CV}(g)$ and $\mathrm{AV}(g \circ f) = g(\mathrm{AV}(f)) \cup \mathrm{AV}(g)$.*

Proof. By the chain rule, $(g \circ f)'(z) = g'\big(f(z)\big)f'(z)$, and thus

$$\begin{aligned} \mathrm{CP}(g \circ f) &= \mathrm{CP}(f) \cup f^{-1}\big(\mathrm{CP}(g)\big), \\ \mathrm{CV}(g \circ f) &= (g \circ f)\big(\mathrm{CP}(g \circ f)\big) \\ &\subseteq (g \circ f)\,\mathrm{CP}(f) \cup g\big(\mathrm{CP}(g)\big) \\ &= g\big(\mathrm{CV}(f)\big) \cup \mathrm{CV}(g). \end{aligned}$$

Observe that the set $f^{-1}\big(\mathrm{CP}(g)\big)$ may be empty, and hence the other inclusion does not hold in general.

Finally, if γ is an asymptotic path of $g \circ f$ with asymptotic value a, then either $f\big(\gamma(t)\big) \to b \in \mathrm{AV}(f)$ as $t \to +\infty$, where $g(b) = a$, or $f(\gamma)$ is an asymptotic path of g and $a \in \mathrm{AV}(g)$. Therefore we have $\mathrm{AV}(g \circ f) \subseteq g\big(\mathrm{AV}(f)\big) \cup \mathrm{AV}(g)$ and the opposite inclusion follows easily. ∎

Let $\mathcal{B}$ and $\mathcal{B}^*$ be the bounded-type classes defined in Section 2.1. Observe that, by Lemma 3.11, both $\mathcal{B}$ and $\mathcal{B}^*$ are closed under composition. Recall that Theorem 3.2 establishes a way to construct functions in $\mathcal{B}^*$ from functions in $\mathcal{B}$. To prove this theorem, we need the following preliminary result.

Proposition 3.12. *Let* $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ *with* $n \in \mathbb{Z}$ *and let* g, h *be non-constant entire functions. If the functions* $f_\infty(z) := z^n \exp\big(g(z)\big)$ *and* $f_0(z) := z^n \exp\big(-h(z)\big)$ *as well as* $1/f_\infty$ *and* $1/f_0$ *have bounded type, then* $f \in \mathcal{B}^*$.

Note that if $n \geqslant 0$, then f_∞ and f_0 are transcendental entire functions, while if $n < 0$, then they are meromorphic functions on $\mathbb{C}$ with a pole at the origin (which is omitted).

Proof of Proposition 3.12. We can express

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big) = f_\infty(z) \cdot \exp\big(h(1/z)\big).$$

Suppose that $f_\infty(z)$ tends to a finite value $a \in \mathbb{C}$ as $z \to \infty$ along an asymptotic path γ. Then $f(z) \to e^{h(0)}a$ as $z \to \infty$ along γ. Conversely, if $f(z)$ tends to a finite value $a \in \mathbb{C}$ as $z \to \infty$ along an asymptotic path γ, then $f(z) \to a/e^{h(0)}$ as $z \to \infty$ along γ. Hence we have $\mathrm{AV}_\infty(f) = e^{h(0)} \cdot \mathrm{AV}(f_\infty)$.

Differentiating f, we obtain

$$f'(z) = f(z)\left(-\frac{h'(1/z)}{z^2} + \frac{f'_\infty(z)}{f_\infty(z)}\right),$$

or, equivalently,

$$\frac{zf'(z)}{f(z)} = -\frac{h'(1/z)}{z} + \frac{zf'_\infty(z)}{f_\infty(z)}.$$

It follows from [EL92, Lemma 1], which is proved by an application of Koebe's 1/4-theorem to the inverse of a logarithmic transform of f (see also Lemma 3.21), that if $f \in \mathcal{B}$, then there is a constant $R_0 > 0$ such that

$$\left|z\frac{f'(z)}{f(z)}\right| \geqslant \frac{1}{4\pi}\big(\log|f(z)| - \log R_0\big) \quad \text{for } z \in D(0, R_0), \tag{3.1}$$

and hence

$$\eta_f := \lim_{R\to+\infty} \inf\left\{\left|z\frac{f'(z)}{f(z)}\right| \;:\; |f(z)| > R\right\} = +\infty. \tag{3.2}$$

If $n < 0$, the function f_∞ is meromorphic but, since the pole at $z = 0$ is omitted and $\operatorname{sing}(f_\infty^{-1})$ is bounded away from the origin, the proof of Lemma 3.21 can be adapted to obtain inequality (3.1) in this case as well. Suppose that f_∞ has bounded type, then

$$\inf\left\{\left|z\frac{f'_\infty(z)}{f_\infty(z)}\right| \;:\; |f_\infty(z)| > R\right\} \to +\infty \quad \text{as } R \to +\infty.$$

Since f_∞ is entire, the components of the set $\{z \in \mathbb{C} \;:\; |f_\infty(z)| > R\}$ are all unbounded and tend to infinity as $R \to +\infty$ in the sense that their distance from the origin tends to infinity. Therefore, since

$$\exp\big(h(1/z)\big) \to \exp\big(h(0)\big) \quad \text{and} \quad \frac{h'(1/z)}{z} \to 0 \quad \text{as } z \to \infty,$$

there exists $M, N > 0$ such that if $|f(z)| > R$ and $|z| \geqslant 1$ then

$$|f_\infty(z)| = \frac{|f(z)|}{\exp\big(\operatorname{Re} h(1/z)\big)} > \frac{R}{M} \quad \text{and} \quad \left|\frac{h'(1/z)}{z}\right| < N,$$

and so, the quantity

$$\inf\left\{\left|z\frac{f'(z)}{f(z)}\right| : |f(z)|>R,\ |z|\geqslant 1\right\} \geqslant \inf\left\{\left|z\frac{f'_\infty(z)}{f_\infty(z)}\right| : |f_\infty(z)|>\frac{R}{M}\right\} - N$$

tends to $+\infty$ as $R \to +\infty$. Hence, $CV(f)$ cannot contain a sequence of critical values whose critical points are in $\mathbb{C} \setminus \mathbb{D}$ that accumulate at infinity, because if $f(z)$ is a critical value, then we have $zf'(z)/f(z) = 0$. Similarly, in a neighbourhood of zero, the quantity

$$\inf\left\{\left|z\frac{f'(z)}{f(z)}\right| : |f(z)| < \frac{1}{R}, \ |z| \leqslant 1\right\} \geqslant \inf\left\{\left|z\frac{f_0'(z)}{f_0(z)}\right| : |f_0(z)| > \frac{R}{M'}\right\} - N'$$

tends to $+\infty$ as $R \to +\infty$, and thus f has no critical values accumulating to zero whose critical points are in $\mathbb{D}$. Finally, since we are assuming that the functions $1/f_\infty$ and $1/f_0$ have bounded type too, $0 \notin \operatorname{sing}(f_\infty^{-1})' \cup \operatorname{sing}(f_0^{-1})'$ and therefore the quantities

$$\inf\left\{\left|z\frac{f'(z)}{f(z)}\right| : |f(z)| < \frac{1}{R}, \ |z| \geqslant 1\right\}, \ \inf\left\{\left|z\frac{f'(z)}{f(z)}\right| : |f(z)| > R, \ |z| \leqslant 1\right\}$$

tend to $+\infty$ as $R \to +\infty$. Hence $f \in \mathcal{B}^*$. ∎

Sixsmith [Six14] showed that if $f \notin \mathcal{B}$, then $\eta_f = 0$, where η_f is the quantity defined in (3.2), and thus provided an alternative characterisation of functions in the class $\mathcal{B}$. This was later generalised by Rempe-Gillen and Sixsmith in [RS15].

Theorem 3.2 states that if g, h are in the class $\mathcal{B}$, then the function $f(z) = \exp\big(g(z) + h(1/z)\big)$ is in class $\mathcal{B}^*$. Thus, it can be used to produce examples of functions in the class $\mathcal{B}^*$ from functions in the class $\mathcal{B}$ (see Example 3.15). Recall that Keen proved that if g, h are polynomials and $n \in \mathbb{Z}$, then $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ is in the class $\mathcal{B}^*$ as well (see Proposition 3.34 and Lemma 3.35).

Proof of Theorem 3.2. Let $f_\infty = \exp \circ g$ where $g \in \mathcal{B}$. By Lemma 3.11,

$$AV(f_\infty) = AV(\exp) \cup \exp(AV(g)) = \exp(AV(g)) \cup \{0\},$$

$$CP(f_\infty) = CP(g) \cup g^{-1}\big(CP(\exp)\big) = CP(g) \cup g^{-1}\big(\emptyset\big) = CP(g),$$

and both $CV(f_\infty) = \exp\big(CV(g)\big)$ and $AV(f_\infty)$ are bounded in $\mathbb{C}$. On the other hand,

$$AV(1/f_\infty) = AV(\exp) \cup \exp(AV(-g)) = \exp(-AV(g)) \cup \{0\},$$

$$CP(1/f_\infty) = CP(-g) = CP(g),$$

and therefore $CV(1/f_\infty) = \exp(-CV(g))$ and $AV(1/f_\infty)$ are bounded in $\mathbb{C}$ too. Similarly, since $h \in \mathcal{B}$, the functions $f_0(z) = \exp(-h(z))$ and $1/f_0$ have bounded type. Therefore f_∞ and f_0 satisfy the hypothesis of Proposition 3.12 and so the function $f(z) = \exp(g(z) + h(1/z))$ is in the class $\mathcal{B}^*$. ∎

Remark 3.13. Observe that if $n \neq 0$ and $f(z) = z^n \exp(g(z))$ with $g \in \mathcal{B}$, then the set $CV(f)$ may accumulate at zero ($n > 0$) or to infinity ($n < 0$) despite the fact that $CV(g)$ is bounded. Thus, Theorem 3.2 is optimal.

Remark 3.14. The converse of Theorem 3.2 is not true in general, as the critical values of g can be unbounded in a vertical band and the critical values of f_∞ be bounded in an annulus. For example, observe that the Fatou function $g(z) = z + 1 + e^{-z}$ is not in the class $\mathcal{B}$, while the function $f(z) = \exp(g(z) + 1/z)$ is in the class $\mathcal{B}^*$ by Proposition 3.12 as $CV(e^g) = \{e^2\}$ and $AV(e^g) = \{0\}$.

Example 3.15. We give a couple of examples of functions in the class $\mathcal{B}^*$ constructed from functions in the class $\mathcal{B}$ using Theorem 3.2.

(i) The function $f(z) = \exp(\sin z/z + 1/z)$ is in the class $\mathcal{B}^*$ and the set $\text{sing}(f^{-1})$ contains infinitely many points that accumulate at $z = 1$.

(ii) The function $f(z) = \exp(\exp z + 1/z)$ is in the class $\mathcal{B}^*$ and has a finite asymptotic value $a = 1$.

3.3 LOGARITHMIC COORDINATES FOR THE CLASS $\mathcal{B}^*$

Let f be a transcendental entire function or a transcendental self-map of $\mathbb{C}^*$. Let $a \in \hat{\mathbb{C}}$ and let $\hat{D}(a, r)$ denote the disc centred at a of radius r in the spherical metric. For $r > 0$, choose $U(r)$ to be a connected component of $f^{-1}(\hat{D}(a, r))$ such that if $0 < r_1 < r_2$, then $U(r_1) \subseteq U(r_2)$. We say that U is a *logarithmic singularity* over a if

$$f : U(r) \to \hat{D}(a, r) \setminus \{a\}$$

is a universal covering for some $r > 0$ (see [Ive14] for a classification of the singularities of the inverse function). Transcendental self-maps of $\mathbb{C}^*$ have logarithmic singularities over both zero and infinity.

Definition 3.16 (Logarithmic tract)**.** Let $f \in \mathcal{B}^*$ and let $A \subseteq \mathbb{C}$ be a topological annulus bounded away from zero and infinity that contains the set $S(f)$. Denote $W := W_0 \cup W_\infty$, where W_0 and W_∞ are the components of $\mathbb{C}^* \setminus A$ whose closure in $\hat{\mathbb{C}}$ contains, respectively, zero and infinity. A *(logarithmic) tract* of f is a connected component of the set $\mathcal{V} = f^{-1}(W_0) \cup f^{-1}(W_\infty)$.

Note that if V is a tract of f, then the map $f : V \to W_i$ is a universal covering, where $i \in \{0, \infty\}$. The following lemma is a well-known classification of the coverings of the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ [Hat02] (see also [For91]). If X is a Riemann surface, we say that two holomorphic coverings $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ of X are *equivalent* if there exists a conformal map $p_{21} : \widetilde{X_2} \to \widetilde{X_1}$ such that $p_2 = p_1 \circ p_{21}$.

Lemma 3.17 (Coverings of $\mathbb{D}^*$)**.** *Let $U \subseteq \hat{\mathbb{C}}$ and let $f : U \to \mathbb{D}^*$ be a holomorphic covering. Then either U is conformally equivalent to $\mathbb{D}^*$ and f is equivalent to z^d, or U is simply connected and f is a universal covering and hence equivalent to the exponential map.*

In particular, the closure of each tract in $\hat{\mathbb{C}}$ contains only one of the essential singularities. Now we are going to introduce a logarithmic change of variables.

Definition 3.18 (Logarithmic coordinates)**.** Let $f \in \mathcal{B}^*$ and consider the sets $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and $H := \exp^{-1}(W) = H_0 \sqcup H_\infty$ where $H_0 = \exp^{-1}(W_0)$ and $H_\infty = \exp^{-1}(W_\infty)$ contain, respectively, a left and a right half-plane. A *logarithmic transform* of f is a continuous function $F : \mathcal{T} \to H$ which makes the following diagram commute.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ \mathcal{V} \arrow[r, "f"'] & W \end{tikzcd}$$

The connected components of $\mathcal{T}$ are called *tracts* of F and can be classified into four types:

$$\mathcal{T} = \mathcal{T}_0^0 \sqcup \mathcal{T}_0^\infty \sqcup \mathcal{T}_\infty^0 \sqcup \mathcal{T}_\infty^\infty,$$

where the lower index indicates if the tracts have zero or infinity in their closure and the upper index indicates if they are mapped to H_0 or H_∞ by F. We define $\mathcal{T}_0 := \mathcal{T}_0^0 \sqcup \mathcal{T}_0^\infty$ and $\mathcal{T}_\infty := \mathcal{T}_\infty^0 \sqcup \mathcal{T}_\infty^\infty$.

In the entire case, often the expressions 'lift' and 'logarithmic transform' are used interchangeably to refer to F which is defined on $\mathcal{T}$ only. In this thesis we reserve the word *lift* for an entire function $\tilde{f}$ such that $\exp \circ \tilde{f} = f \circ \exp$.

Remark 3.19. Observe that we can obtain F as the restriction of a lift $\tilde{f}$ of f to the set $\mathcal{T}$. However, since F is only defined on $\mathcal{T}$, we can add a different integer multiple of $2\pi i$ to F on each tract T, and hence F is not necessarily the restriction of a transcendental entire function $\tilde{f}$.

Theorem 3.20. *If $f \in \mathcal{B}^*$, then a logarithmic transform $F : \mathcal{T} \to H$ of f satisfies the following properties:*

(a) the set H is the union of two disjoint $2\pi i$-periodic Jordan domains H_0 and H_∞ containing, respectively, a left and a right half-plane;

(b) every component of $\mathcal{T}$ is an unbounded Jordan domain whose points have real part either bounded from below and unbounded from above or unbounded from below and bounded from above;

(c) the components of $\mathcal{T}$ have disjoint closures and accumulate only at zero and infinity;

(d) for every component T of $\mathcal{T}$, the function $F_{|T} : T \to H$ is a conformal isomorphism;

(e) for every component T of $\mathcal{T}$, the function $\exp_{|T}$ is injective;

(f) the set $\mathcal{T}$ is invariant under translation by $2\pi i$.

Moreover, there exists a curve $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ joining zero to infinity, where $\mathcal{V} = \exp \mathcal{T}$.

Proof. These properties follow easily from the fact that the exponential map is a holomorphic cover and, in particular, a local homeomorphism. The fact that there exists a curve $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ joining zero to infinity is a straightforward consequence of (b) and (c) in the case that $\mathcal{V}$ consists of finitely many tracts. Otherwise, this follows from Carathéodory's theorem and the fact that $\mathcal{V}$ is locally connected (see [BF15, Lemma 2.1]). Hence, we can define a continuous branch of the logarithm on $\mathbb{C}^* \setminus \delta$. ∎

We denote by $\mathcal{B}^*_{\log}$ the class of holomorphic functions $F : \mathcal{T} \to H$ satisfying properties (a) to (f) in Theorem 3.20, regardless of whether

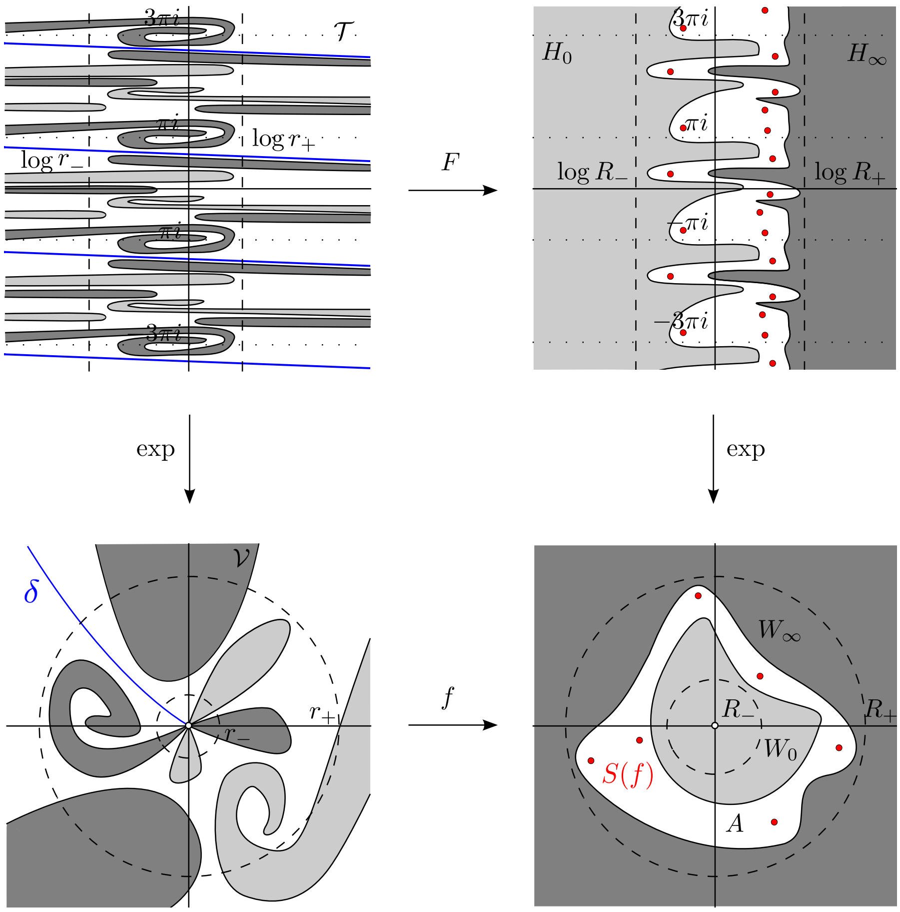


Figure 5: Logarithmic coordinates for a function $f \in \mathcal{B}^*$.

or not they come from a function $f \in \mathcal{B}^*$. The main advantage of working in the class $\mathcal{B}_{\log}$ from [RRRS11] or, in our case, the class $\mathcal{B}^*_{\log}$, is that such functions satisfy the following *expansivity property* (3.3) which implies that points in $I(f)$ eventually escape at an exponential rate.

Lemma 3.21. *Let* $F : \mathcal{T} \to H$ *be a function in the class* $\mathcal{B}^*_{\log}$*. There exists* $R > 0$ *sufficiently large such that if* $|\text{Re } F(z)| \geqslant R$*, then*

$$|F'(z)| \geqslant \frac{1}{4\pi}|\text{Re } F(z)| - R.$$

In particular, there exists $R_0 = R_0(F) > 0$ *so that*

$$|F'(z)| \geqslant 2 \quad \textit{for all } z \in \mathcal{T} \textit{ such that } |\text{Re } F(z)| \geqslant R_0. \tag{3.3}$$

See [EL92, Lemma 1] for the original result for entire functions. The proof relies on properties (a), (d) and (e) of logarithmic transforms, which are common in both settings, and Koebe's 1/4-theorem.

Sullivan [Sul85] proved that rational maps have no wandering domain. Following this result, Keen [Kee88], Kotus [Kot87] and Makienko [Mak87] proved independently that transcendental self-maps of $\mathbb{C}^*$ with finitely many singular values have no wandering domains. In [Kot87], Kotus also showed that finite-type maps in $\mathbb{C}^*$ have no Baker domains. Here we show that bounded-type transcendental self-maps of $\mathbb{C}^*$ have no escaping Fatou components by adapting the proof that Eremenko and Lyubich gave for the class $\mathcal{B}$ [EL92, Theorem 1].

Proof of Theorem 3.1. Suppose to the contrary that there exists a point $z_0 \in F(f) \cap I(f)$. Then, by normality, there exists some $R_0 > 0$ so that $B_0 := B(z_0, R_0) \subseteq F(f) \cap\ I(f)$. Since $B_0 \subseteq I(f)$, there exists $N_0 \in \mathbb{N}_0$ such that the sets $B_n := f^n(B_0)$, $n \in \mathbb{N}$, are contained in the set of tracts $\mathcal{V}$ of f for all $n \geqslant N_0$; we can assume without loss of generality that $N_0 = 0$. Let C_0 be a connected component of $\exp^{-1}(B_0)$ and put $C_n := F^n(C_0)$ for $n \in \mathbb{N}$. For every $R > 0$, there exists $N = N(R) \in \mathbb{N}_0$ such that

$$C_n \subseteq \{z \in \mathbb{C} \;:\; |\operatorname{Re} z| > R\} \quad \text{for all } n > N.$$

Take any point $\zeta_0 \in C_0$ and, for all $n > 0$, set $\zeta_n := F^n(\zeta_0) \in C_n$ and $d_n := \operatorname{dist}(\zeta_n, \partial C_n)$. Then Koebe's 1/4-theorem implies that

$$d_{n+1} \geqslant \frac{1}{4} d_n |F'(\zeta_n)| \quad \text{for all } n \in \mathbb{N}.$$

Since $|\operatorname{Re} F(\zeta_n)| \to +\infty$ as $n \to \infty$, by Lemma 3.21, we have $|F'(\zeta_n)| \to +\infty$ and hence $d_n \to +\infty$. But this contradicts property (e) of functions in the class $\mathcal{B}^*_{\log}$ because $\mathcal{T}$ does not contain any vertical segment of length 2π. Thus $F(f) \cap I(f) = \emptyset$ and $I(f) \subseteq J(f)$. ■

By property (a) in Theorem 3.20, if $F : \mathcal{T} \to H$ is in the class $\mathcal{B}^*_{\log}$, then the set H contains the union of two half-planes of the form

$$\mathbb{H}^{\pm}_R := \{z \in \mathbb{C} \;:\; |\operatorname{Re} z| > R\} = \mathbb{H}^-_R \sqcup \mathbb{H}^+_R$$

for some $R > 0$. We call F *normalised* if $H = \mathbb{H}^{\pm}_R$ for some $R > 0$ and the function F satisfies the expansivity property (3.3).

Definition 3.22 (Normalisation)**.** We say that a logarithmic transform $F : \mathcal{T} \to H$ in $\mathcal{B}^*_{\log}$ is *normalised* if $\overline{\mathcal{T}} \cap \{z \in \mathbb{C} \ : \ \operatorname{Re} z = 0\} = \emptyset$, the set $H = \mathbb{H}^{\pm}_R$ for some $R > 0$, and the expansivity property (3.3) is satisfied in H. We denote this class of functions by $\mathcal{B}^{*n}_{\log}$.

Logarithmic transforms of transcendental entire functions can be normalised so that H is the right half-plane $\mathbb{H}$. In contrast, in the punctured plane, when we say that F is normalised we need to specify the constant R. The next lemma shows that we can always assume that F is in the class $\mathcal{B}^{*n}_{\log}$ by restricting the function to a smaller set.

Lemma 3.23. *Let $F : \mathcal{T} \to H$ be a function in the class $\mathcal{B}^*_{\log}$. There exists a constant $R = R(F) > 0$ such that $\mathbb{H}^{\pm}_R \subseteq H$ and the restriction of F to $F^{-1}(\mathbb{H}^{\pm}_R)$ is a normalised logarithmic transform.*

Proof. Suppose that F is not normalised. Let $\{B_n\}_{n\in\mathbb{Z}}$ denote the connected components of the set $\mathbb{C} \setminus \exp^{-1}(\delta)$, where δ is the curve from Theorem 3.20. For $n \in \mathbb{Z}$, the sets

$$X_n = \mathcal{T}_0 \cap B_n \cap \mathbb{H}^+ \quad \text{and} \quad Y_n = \mathcal{T}_\infty \cap B_n \cap \mathbb{H}^-$$

are bounded and hence their images $F(X_n)$ and $F(Y_n)$ have bounded real part. All the sets $F(X_n)$ and $F(Y_n)$, $n \in \mathbb{Z}$, are vertical translates of $F(X_0)$ and $F(Y_0)$ and hence $F(\mathcal{T}_0 \cap \mathbb{H}^+)$ and $F(\mathcal{T}_\infty \cap \mathbb{H}^-)$ have bounded real part. Therefore, there exists $R_1 > 0$ sufficiently large such that

$$\big(F(\mathcal{T}_0 \cap \mathbb{H}^+) \cup F(\mathcal{T}_\infty \cap \mathbb{H}^-)\big) \cap \mathbb{H}^{\pm}_{R_1} = \emptyset.$$

Then, if $R_0 = R_0(F) > 0$ is the constant from Lemma 3.21 so that $|F'(z)| > 2$ if $|\operatorname{Re} F(z)| \geqslant R_0$, it is enough to put $R := \max\{R_0, R_1\}$. ■

The following lemma is a stronger version of the expansivity property (3.3) for functions in $\mathcal{B}^{*n}_{\log}$, and says that escaping orbits eventually separate at an exponential rate. The proof of [RRRS11, Lemma 3.1] can be adapted easily to prove this lemma.

Lemma 3.24. *Let $F : \mathcal{T} \to H$ be in the class $\mathcal{B}^{*n}_{\log}$ with $H = \mathbb{H}^{\pm}_R$ for some $R > 0$. If T is a tract of F and $z, w \in T$ are such that $|z - w| \geqslant 8\pi$, then*

$$|F(z) - F(w)| \geqslant \exp\left(\frac{|z-w|}{8\pi}\right) \cdot \big(\min\{|\operatorname{Re} F(z)|, \ |\operatorname{Re} F(w)|\} - R\big).$$

Next we introduce a subclass of $\mathcal{B}^*_{\log}$ consisting of the functions $F : \mathcal{T} \to H$ for which the image $F(\mathcal{T})$ covers the whole $\overline{\mathcal{T}}$, which have nicer properties.

Definition 3.25 (Disjoint type)**.** We say that a function $F : \mathcal{T} \to H$ in the class $\mathcal{B}^*_{\log}$ is of *disjoint type* if $\overline{\mathcal{T}} \subseteq H$.

If $f \in \mathcal{B}^*$ and $A = \mathbb{C}^* \setminus W$ is an annulus containing $S(f)$, then $f(\mathbb{C}^* \setminus \mathcal{V}) \subseteq A$, where $\mathcal{V} = f^{-1}(W)$. In the case that f has a logarithmic transform F that is of disjoint type (with $H = \exp^{-1}(W)$), we have $A \subseteq \mathbb{C}^* \setminus \mathcal{V}$ and $f(A) \subseteq A$. Hence $A \subseteq F(f)$ and it follows from the classification of Fatou components that, in this situation, $F(f)$ consists of a single doubly connected component U which is the immediate basin of attraction of an attracting fixed point in A.

Remark 3.26. Independently of [RRRS11], Barański showed that the Julia set of bounded-type maps in the class $\mathcal{B}$ consists of disjoint hairs that are homeomorphic to $[0, +\infty)$ (we call them dynamic rays) and that the endpoints of these hairs are the only points in $J(f)$ accessible from $F(f)$ [Bar07, Theorem C].

Example 3.27. The function $f(z) = \exp\big(0.3(z + 1/z)\big)$ is in the class $\mathcal{B}^*$ and has a logarithmic transform of disjoint type (see Figure 6).

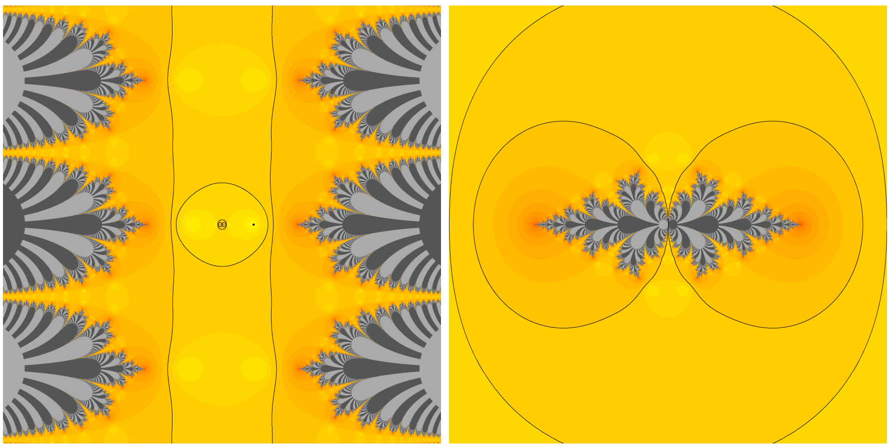


Figure 6: Phase space of the function $f(z) = \exp\big(0.3(z + 1/z)\big)$ which has a disjoint-type logarithmic transform (see Example 3.27). In orange, the basin of attraction of the fixed point $z_0 \simeq 2.2373$. Scale: $z \in [-16, 16] + i[-16, 16]$ (left), $z \in [-0.3, 0.3] + i[-0.3, 0.3]$ (right).

Sometimes tracts exhibit better geometric properties that make them easier to study. In the next section we will see that this is the case for transcendental self-maps of $\mathbb{C}^*$ of finite order.

Definition 3.28 (Good geometry properties)**.** Let $F \in \mathcal{B}^*_{\log}$ and let T be a tract of F.

(a) We say that T has *bounded wiggling* if there exist $K > 1$ and $\mu > 0$ such that for every $z_0 \in \overline{T}$, every point z on the hyperbolic geodesic of T that connects z_0 to ∞ satisfies

$$|\operatorname{Re} z| > \frac{1}{K}|\operatorname{Re} z_0| - \mu.$$

In the case $K = 1$ and $\mu = 0$ we say that T has *no wiggling*. A function $F \in \mathcal{B}^*_{\log}$ has *uniformly bounded wiggling* if the wiggling of all tracts of F is bounded by the same constants K, μ.

(b) We say that T has *bounded slope* if there exist constants $\alpha, \beta > 0$ such that

$$|\operatorname{Im} z - \operatorname{Im} w| \leqslant \alpha \max\{|\operatorname{Re} z|, |\operatorname{Re} w|\} + \beta$$

for all $z, w \in T$. Equivalently, T contains a curve $\gamma : [0, \infty) \to T$ such that $|F(\gamma(t))| \to \pm\infty$ and

$$\limsup_{t\to\infty} \frac{|\operatorname{Im} \gamma(t)|}{|\operatorname{Re} \gamma(t)|} < \infty.$$

We say that T has *zero slope* if this limit is zero.

We say F has *good geometry* if the tracts of F have bounded slope and uniformly bounded wiggling.

Remark 3.29. (i) Observe that it is enough that a tract T from $\mathcal{T}_\alpha$, $\alpha \in \{0, \infty\}$, has bounded slope to ensure that all tracts in $\mathcal{T}_\alpha$ do. We can use the same constants (α, β) for $\mathcal{T}_\infty$ and $\mathcal{T}_0$: if they have bounded slope with different values (α_1, β_1) and (α_2, β_2) it is enough to take $\alpha := \max\{\alpha_1, \alpha_2\}$ and $\beta := \max\{\beta_1, \beta_2\}$.

(ii) If $F, G \in \mathcal{B}^{*n}_{\log}$ and G has bounded slope, then $G \circ F$ has bounded slope with the same constants as G.

3.4 ORDER OF GROWTH IN $\mathbb{C}^*$

Recall that the *order* of an entire function is defined to be the infimum of $\rho \in \mathbb{R} \cup \{\infty\}$ such that $\log|f(z)| = \mathcal{O}(|z|^\rho)$ as $z \to \infty$. Equivalently,

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

where

$$M(r,f) := \max_{|z|=r} |f(z)| < +\infty.$$

Polynomials have order zero and the function $\exp(z^k)$, $k \in \mathbb{N}$, has order k. There are also transcendental entire functions of order zero and of infinite order.

When we deal with holomorphic self-maps of $\mathbb{C}^*$, controlling the growth requires us to study how $|f(z)|$ tends to zero or infinity when z approaches zero or infinity. Observe that if f is such map, then $1/f$ is also holomorphic on $\mathbb{C}^*$, and

$$m(r,f) := \min_{|z|=r} |f(z)| = \frac{1}{M(r,1/f)} > 0.$$

As before, for simplicity, we will write $M(r)$ and $m(r)$ when it is clear what the function f is.

A priori, the notion of order of growth in this context involves the following four quantities:

$$\rho^\infty_{\max}(f) := \limsup_{r\to+\infty} \frac{\log\log M(r)}{\log r},\ \rho^\infty_{\min}(f) := \limsup_{r\to+\infty} \frac{\log(-\log m(r))}{\log r},$$

$$\rho^0_{\max}(f) := \limsup_{r\to 0} \frac{\log\log M(r)}{-\log r},\ \rho^0_{\min}(f) := \limsup_{r\to 0} \frac{\log(-\log m(r))}{-\log r}.$$

However, if an entire function f has no zeros, then $\rho(f) = \rho(1/f)$ as a consequence of the fact that you can write the order in terms of the Nevanlinna characteristic function $T(R,f)$:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and Jensen's formula says that

$$T(r,f) = T(r,1/f) + \log|f(0)|$$

(see section 1.2 of [Hay64]). It follows from the general expression of a transcendental self-map of $\mathbb{C}^*$

$$f(z) = z^n \exp(g(z) + h(1/z)),$$

with $n \in \mathbb{Z}$ and g, h non-constant entire functions that

$$\log|f(z)| = n \log|z| + \operatorname{Re} g(z) + \operatorname{Re} h(0) + o(1) \quad \text{as } z \to \infty,$$

and therefore

$$\log M(r, f) = \log M(r, e^g) + O(\log r) \quad \text{as } z \to \infty. \tag{3.4}$$

Note that in a neighbourhood of infinity the term $h(1/z)$ is not relevant and the same happens with $g(z)$ in a neighbourhood of the origin. Then, putting (3.4) into the four order quantities defined above and using Jensen's formula we obtain

$$\rho^{\infty}_{\max}(f) = \rho^{\infty}_{\max}(e^g) = \rho(e^g) = \rho^{\infty}_{\min}(e^g) = \rho^{\infty}_{\min}(f) \tag{3.5}$$

and, similarly, at zero

$$\rho^{0}_{\max}(f) = \rho^{\infty}_{\max}(e^h) = \rho(e^h) = \rho^{\infty}_{\min}(e^h) = \rho^{0}_{\min}(f),$$

so, in fact, the order of growth of f involves only two quantities.

Definition 3.30 (Order of growth)**.** Let f be a transcendental self-map of $\mathbb{C}^*$ of the form

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big)$$

with $n \in \mathbb{Z}$ and g, h non-constant entire functions. We say that f has *finite order* if both quantities

$$\rho_\infty(f) := \rho(e^g) \quad \text{and} \quad \rho_0(f) := \rho(e^h)$$

are finite.

Example 3.31. The functions $f(z) = z^n \exp(P(z) + Q(1/z))$, with $n \in \mathbb{Z}$ and P, Q polynomials, are transcendental self-maps of $\mathbb{C}^*$ of finite order and $\rho_\infty(f) = \deg P$ and $\rho_0(f) = \deg Q$.

Remark 3.32. Keen [Kee88] defined the order of transcendental self-maps of $\mathbb{C}^*$ using

$$\widetilde{M}(r,f) = \max_{z\in\partial A_r} |f(z)| \quad \text{and} \quad \widetilde{m}(r,f) = \min_{z\in\partial A_r} |f(z)|$$

for $r > 0$, where $A_r := \{z \in \mathbb{C} \ : \ 1/r < |z| < r\}$. It follows from the maximum principle that $\widetilde{M}(r,f)$ and $\widetilde{m}(r,f)$ are respectively the maximum and minimum of $|f(z)|$ in the whole annulus A_r (in the same way that, for an entire function, we have $M(r) = \max_{z\in D(0,r)} |f(z)|$). In our notation,

$$\widetilde{M}(r,f) = \max\{M(r),\ M(1/r)\} \quad \text{and} \quad \widetilde{m}(r,f) = \min\{m(r),\ m(1/r)\}.$$

Next we will see that, in fact, every holomorphic self-map of $\mathbb{C}^*$ that has finite order necessarily has to be of the form given in Example 3.31. We will begin by stating a classical result concerning entire functions of finite order due to Pólya [Pól25]; see also [Hay64, Theorem 2.9].

Lemma 3.33. *If f is a non-constant entire function of finite order with no zeros, then $f(z) = \exp(h(z))$ and h is a polynomial.*

Using Lemma 3.33, we obtain the following.

Proposition 3.34. *Every transcendental self-map of $\mathbb{C}^*$ of finite order is of the form*

$$f(z) = z^n \exp(P(z) + Q(1/z))$$

for some $n \in \mathbb{Z}$ and $P, Q \in \mathbb{C}[z]$.

Keen proved the stronger result that every topological conjugacy class of analytic self-maps of $\mathbb{C}^*$ contains a function of this form [Kee89, Theorem 1], but we give a direct proof of Proposition 3.34 for completeness.

Proof. We know that every transcendental self-map of $\mathbb{C}^*$ is of the form

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big)$$

for some $n \in \mathbb{Z}$ and g, h non-constant entire functions. Thus, by (3.5),

$$\rho(e^g) = \rho_\infty(f) < +\infty$$

and so it follows from Lemma 3.33 that g has to be a polynomial. On the other hand,

$$\rho(e^h) = \rho_0(f) < +\infty$$

and so h has to be a polynomial as well. ∎

Keen also showed that, in $\mathbb{C}^*$, finite order implies finite type [Kee89, Proposition 2]. This is very different to what happens for the entire case, where we have functions of finite order in the class $\mathcal{B}$ that are not in the *Speiser class* $\mathcal{S}$ of finite-type transcendental entire functions. An example of such a function is given by $\sin(z)/z$ which has order one and infinitely many critical values in any open interval in $\mathbb{R}$ containing the origin. We state Keen's result for future reference.

Lemma 3.35. *Let f be a transcendental self-map of $\mathbb{C}^*$. If f has finite order with $\rho_\infty(f) = p$ and $\rho_0(f) = q$, then $\operatorname{sing}(f^{-1})$ consists of at most $p+q$ critical values together with the asymptotic values zero and infinity.*

Finally, we show that the tracts of finite order functions have a fairly simple geometry.

Proposition 3.36. *Let f be a transcendental self-map of $\mathbb{C}^*$ of finite order and let $F \in \mathcal{B}^{*n}_{\log}$ be a logarithmic transform of f. Then f has a finite number of tracts and the tracts of F have zero slope and can be chosen to have no wiggling.*

Proof. Suppose that $\rho_\infty(f) = p$ and $\rho_0(f) = q$ with $p, q \geqslant 1$. Then, by Proposition 3.34,

$$f(z) = z^n \exp\big(P(z) + Q(1/z)\big),$$

where $n \in \mathbb{Z}$ and P, Q are, respectively, polynomials of degree p, q. We focus on the tracts whose closure in $\hat{\mathbb{C}}$ contains infinity; the case where the closure contains zero is similar. We have

$$|f(z)| = \exp\big(\operatorname{Re}(az^p) + o(\operatorname{Re}(z^p))\big) \quad \text{as } z \to \infty, \tag{3.6}$$

where $a \in \mathbb{C}^*$. Let $\phi = \arg(a)$. For large values of R, the tracts of f defined by $|f(z)| > R$ are contained in the sectors determined by the preimages of the imaginary axis by the map az^p, that is, the radial lines of angle $(k\pi + \pi/2 - \phi)/p$, $k \in \mathbb{Z}$. Tracts that map to a neighbourhood of infinity lie in the sectors containing the radial lines of angle $(2k\pi - \phi)/p$, $0 \leqslant k < p$, while tracts that map to a neighbourhood of zero lie in the sectors containing the radial lines of angle $((2k+1)\pi - \phi)/p$, $0 \leqslant k < p$. The preimages of radial lines by the exponential function are horizontal lines and hence the tracts of F are contained in horizontal bands and have zero slope.

Finally, since the boundaries of the tracts tend asymptotically to these horizontal lines, the tracts of F can be chosen to have no wiggling if R is sufficiently large. ∎

It follows from Proposition 3.34 that, in the punctured plane, functions of finite order (as well as entire functions of finite order with no zeros) can only have integer orders $\rho_0(f)$ and $\rho_\infty(f)$. There are always exactly $2\rho_\infty(f)$ asymptotic paths to infinity corresponding, asymptotically, to the preimages of the positive (asymptotic value infinity) or negative (asymptotic value zero) real line by z^d where $d = \rho_\infty(f)$. Therefore the asymptotic paths alternate as you go around a circle of large radius (see Figure 7). Similarly, in a neighbourhood of zero there are $2\rho_0(f)$ asymptotic paths with the same structure. Each of these asymptotic paths is contained in a logarithmic tract and vice versa.

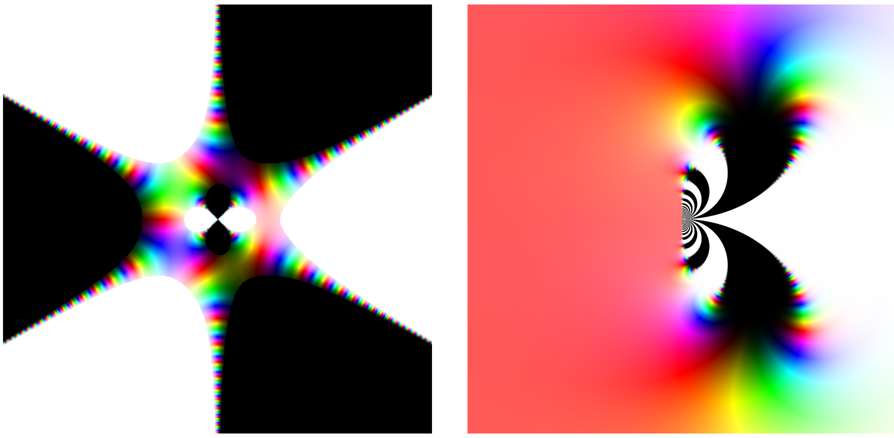


Figure 7: Logarithmic tracts of functions of finite order with $\rho_\infty(f) = 3$ and $\rho_0(f) = 2$ (left) and infinite order (right). The colour of every point $z \in \mathbb{C}^*$ has been chosen according to the modulus (luminosity) and argument (hue) of $f(z)$.

Another basic property of entire functions in the class $\mathcal{B}$ is that they have lower order greater than or equal to $1/2$ [Hei48] (see also [RS05a, Lemma 3.5]). This is due to the fact that f is bounded on a path δ to infinity. Note that δ can be chosen to be any path that lies in the complement of the set of tracts of f. Recall that the lower order of an entire function is

$$\lambda(f) := \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

If f is a transcendental self-map of $\mathbb{C}^*$ we consider

$$\lambda_\infty(f) := \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_0(f) := \liminf_{r\to 0} \frac{\log\log 1/m(r,f)}{\log 1/r}.$$

Recall that Theorem 3.3 in Section 3.1 states that, in this setting, $\lambda_0(f) = \rho_0(f)$ and $\lambda_\infty(f) = \rho_\infty(f)$. To prove this, we shall use the Borel-Carathéodory theorem in the form given in [Val49, Theorem 8].

Lemma 3.37 (Borel-Carathéodory theorem)**.** *Let f be a transcendental entire function and define, for $r > 0$,*

$$B(r,f) := \min_{|z|=r} \operatorname{Re} f(z), \quad A(r,f) := \max_{|z|=r} \operatorname{Re} f(z).$$

Then, there is $r_0 = r_0(f) > 0$ and $C = C(f) > 0$ such that

$$B(r) \leqslant M(r) < \frac{R}{R-r}\big(4A(R) + C\big)$$

for all $R > r > r_0$.

Proof of Theorem 3.3. Let $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ with $n \in \mathbb{Z}$ and g, h non-constant entire functions. We treat separately the cases where the function f has finite order and infinite order. For simplicity we only consider $\rho_\infty(f)$ and $\lambda_\infty(f)$; the proof for $\rho_0(f)$ and $\lambda_0(f)$ is completely analogous.

Suppose that $\rho_\infty(f) = p < +\infty$. Then, by Proposition 3.34, g is a polynomial and, by (3.6),

$$\lambda_\infty(f) = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log A(r,g)}{\log r}.$$

Since ar^p, $a > 0$, is an increasing function for $r > 0$, it is clear that $\lambda_\infty(f) = \rho_\infty(f)$.

Now suppose that $\rho_\infty(f) = +\infty$. We use Lemma 3.37 with $R = 2r$: there is $C > 0$ and $r_0 > 0$ such that

$$M(r,g) < 2\big(4A(2r,g) + C\big) \quad \text{for all } r > r_0.$$

Therefore, since g is a transcendental entire function, we have

$$\liminf_{r\to+\infty} \frac{\log A(r,g)}{\log r} \geqslant \liminf_{r\to+\infty} \frac{\log M(r/2,g)}{\log r} = \lim_{r\to+\infty} \frac{\log M(r,g)}{\log r} = +\infty$$

and so $\lambda_\infty(f) = +\infty$. ∎

Observe that if $F \in \mathcal{B}^*_{\log}$, then the tracts of F in each of the sets $\mathcal{T}_0$ and $\mathcal{T}_\infty$ can be ordered with respect to the vertical position around infinity. Therefore it makes sense to speak about a tract being in between two other tracts. This ordering is known as the *lexicographic order* and we will come back to it later (see Definition 3.45).

3.5 SYMBOLIC DYNAMICS AND COMBINATORICS

Maps in the class $\mathcal{B}^*_{\log}$ are defined on a set $\mathcal{T}$, which is a union of tracts, and, therefore, the orbits of some points in $\mathcal{T}$ are truncated if $F^k(z) \notin \mathcal{T}$ for some $k \in \mathbb{N}$. We denote by $J(F)$ the set of points that can be iterated infinitely many times by F.

Definition 3.38 (Julia set of F)**.** Let $F : \mathcal{T} \to H$ be a map in class $\mathcal{B}^*_{\log}$. We define the *Julia set* of F to be

$$J(F) := \{z \in \overline{\mathcal{T}} \ : \ F^n(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \in \mathbb{N}_0\},$$

and, for $K > 0$, we put

$$J^K(F) := \{z \in \overline{\mathcal{T}} \ : \ |\operatorname{Re} F^n(z)| \geqslant K \text{ for all } n \in \mathbb{N}_0\}.$$

As we will see in the following lemma, the reason why $J(F)$ is called the Julia set of F is that points in $J(F)$ project to points in $J(f)$ by the exponential map. However, note that in the case that $F \in \mathcal{B}^*_{\log}$ is a logarithmic transform of a function $f \in \mathcal{B}^*$, there exists an entire function $\tilde{f}$ that is a lift of f, and then $J(F) \subseteq J(\tilde{f}) = \exp^{-1} J(f)$ by a result of Bergweiler [Ber95].

Lemma 3.39. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let $F \in \mathcal{B}^*_{\log}$ be a logarithmic transform of f. If $F \in \mathcal{B}^{*n}_{\log}$, then $\exp J(F) \subseteq J(f)$ and, if F is of disjoint type, then $\exp J(F) = J(f)$.*

Proof. Suppose to the contrary that $z_0 \in \exp J(F) \cap F(f) \neq \emptyset$. Then, proceeding as in the proof of Theorem 3.1, we get a contradiction between the expansivity of F given by (3.3) and the fact that $\mathcal{T}$ does not contain vertical segments of length 2π. Note that in the normalised case we use the expansivity of F with respect to the Euclidean metric, that is, $|F'(z)| \geqslant 2$ for all $z \in \mathcal{T}$ (see Lemma 3.21), while in the disjoint-type case we use the expansivity with respect to the hyperbolic metric on H because $\mathcal{T}$ is compactly contained in H.

If F is of disjoint type, the inclusion $J(f) \subseteq \exp J(F)$ follows from the fact that $f(\mathbb{C}^* \setminus \mathcal{V}) \subseteq A$ and hence $F(f)$ consists of the immediate basin of attraction of an attracting fixed point in $\mathbb{C}^* \setminus \mathcal{V}$, and so

$$J(f) = \mathbb{C}^* \setminus \bigcup_{n \in \mathbb{N}} f^{-n}(\mathbb{C}^* \setminus \mathcal{V})$$

as required. ∎

Recall that in Definition 2.1 we defined the *essential itinerary* of a point $z \in I(f)$ to be the symbol sequence $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ such that

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

for all $n \in \mathbb{N}_0$.

We now introduce the escaping set for a map F in the class $\mathcal{B}^*_{\log}$, which is a subset of the Julia set of F.

Definition 3.40 (Escaping set of F)**.** Let $F : \mathcal{T} \to H$ be a map in the class $\mathcal{B}^*_{\log}$. We define the *escaping set* of F to be

$$I(F) := \{z \in J(F) \; : \; \lim_{n\to\infty} |\operatorname{Re} F^n(z)| = +\infty\} = J(F) \cap \exp^{-1} I(f).$$

In terms of F, a point $z \in I(F)$ has *essential itinerary* $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ if $\operatorname{Re} F^n(z) \leqslant 0$ if and only if $e_n = 0$ for all $n \in \mathbb{N}_0$.

Observe that $\exp I(F) \subseteq I(f)$ and, in fact, every point in $I(f)$ eventually enters $\exp I(F)$. As with $J(F)$, if f is a transcendental self-map of $\mathbb{C}^*$ and $\tilde{f}$ is a lift of f, then $I(F) \subseteq I(\tilde{f})$ but, in general, these sets are different, as $\tilde{f}$ may have points that escape in the imaginary direction which correspond to bounded orbits for f.

For every function $F \in \mathcal{B}^*_{\log}$, we denote by $\mathcal{A}$ (respectively $\mathcal{A}^0_0, \mathcal{A}^\infty_0$, $\mathcal{A}^0_\infty, \mathcal{A}^\infty_\infty$) the *symbolic alphabet* consisting of all tracts in $\mathcal{T}$ (respectively $\mathcal{T}^0_0, \mathcal{T}^\infty_0, \mathcal{T}^0_\infty, \mathcal{T}^\infty_\infty$; see Definition 3.18). We associate a symbol sequence

$(T_n) \in \mathcal{A}^{\mathbb{N}_0}$ to each point $z \in J(F)$ that describes to which tract the iterate $F^n(z)$ belongs for all $n \in \mathbb{N}_0$.

Definition 3.41 (External address of F)**.** Let $F \in \mathcal{B}^*_{\log}$ and let $z \in J(F)$. We define the *external address* of z, $\operatorname{addr}_F(z)$, to be the symbol sequence $\underline{s} = (T_n) \in \mathcal{A}^{\mathbb{N}_0}$ such that $F^n(z) \in \overline{T_n}$ for all $n \in \mathbb{N}_0$.

Remark 3.42. Let F be a normalised logarithmic transform. Then the Bernoulli shift map $\sigma : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0}$ mapping the external address (T_n) to (T_{n+1}) is a *subshift of finite type* on the set

$$\mathcal{A}^{\mathbb{N}_0} = (\mathcal{A}_0^\infty \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}_\infty^\infty \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}_0^0 \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}_\infty^0 \times \mathcal{A}^{\mathbb{N}}),$$

where, if $e_0, e_1 \in \{0, \infty\}$, the set $\mathcal{A}_{e_0}^{e_1} \times \mathcal{A}^{\mathbb{N}}$ consists of the sequences in $\mathcal{A}^{\mathbb{N}_0}$ whose first symbol is in $\mathcal{A}_{e_0}^{e_1}$. Observe that the transition graph of σ is

$$\begin{tikzcd}
\mathcal{A}_0^\infty \times \mathcal{A}^{\mathbb{N}} \arrow[r] \arrow[dr, shift left] & \mathcal{A}_\infty^\infty \times \mathcal{A}^{\mathbb{N}} \arrow[loop right] \arrow[d] \\
\mathcal{A}_0^0 \times \mathcal{A}^{\mathbb{N}} \arrow[loop left] \arrow[u] & \mathcal{A}_\infty^0 \times \mathcal{A}^{\mathbb{N}} \arrow[l] \arrow[ul, shift left]
\end{tikzcd}$$

and, in particular, not all sequences in $\mathcal{A}^{\mathbb{N}_0}$ are external addresses of points in $J(F)$.

We now introduce the notion of admissible external address. Only admissible external addresses can be the external address of a point in $J(F)$.

Definition 3.43 (Admissible external address)**.** We say that an external address $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ is *admissible* if $\underline{s}$ belongs to the set

$$\boldsymbol{\Sigma}_e := \prod_{n \in \mathbb{N}} \mathcal{A}_{e_n}^{e_{n+1}} = \{(T_n) \ : \ T_n \in \mathcal{A}_{e_n}^{e_{n+1}} \text{ for all } n \in \mathbb{N}_0\},$$

for some $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$. In this case, we say that the external address $\underline{s}$ has *essential itinerary* e. We denote by $\boldsymbol{\Sigma}$ the set of *all* admissible external addresses.

Note that, if we define

$$\mathcal{A}_0 := \mathcal{A}_0^0 \sqcup \mathcal{A}_0^\infty \quad \text{and} \quad \mathcal{A}_\infty := \mathcal{A}_\infty^0 \sqcup \mathcal{A}_\infty^\infty$$

then an external address $\underline{s} = (T_n) \in \boldsymbol{\Sigma}$ has essential itinerary $e = (e_n)$ provided that $T_n \in \mathcal{A}_0$ if and only if $e_n = 0$. In terms of essential itineraries, the corresponding transition graph is the complete graph on two vertices,

$$\begin{tikzcd} \mathcal{A}_0 \times \mathcal{T}^{\mathbb{N}} \arrow[loop left] \arrow[r, shift left] & \mathcal{A}_\infty \times \mathcal{T}^{\mathbb{N}} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}.$$

If $F \in \mathcal{B}^{*n}_{\log}$, then $z \in I(F)$ has essential itinerary e if and only if $\operatorname{addr}(z)$ has essential itinerary e. However, if F is not normalised, these two sequences may be different for a certain number of iterates (see Lemma 3.59).

For every admissible external address, we introduce the set of points that have that external address. Note that sometimes we use the term external address to denote a general sequence in $\boldsymbol{\Sigma}$, without being necessarily the external address of any point $z \in J(F)$. Therefore, some of the following sets may be empty. In Definition 2.2, for $e \in \{0, \infty\}^{\mathbb{N}_0}$, we defined $I^{0,0}_e(f)$ to be the set of escaping points whose essential itinerary is *exactly* e and we defined $I_e(f)$ to be the set of escaping points whose essential itinerary is *eventually* a shift of e.

Definition 3.44 (Subsets of $J(F)$). Let F be a function in the class $\mathcal{B}^*_{\log}$. For $\underline{s} \in \boldsymbol{\Sigma}$ and $K > 0$, we define the sets

$$J_{\underline{s}}(F) := \{z \in J(F) \ : \ \operatorname{addr}_F(z) = \underline{s}\},$$

$J^K_{\underline{s}}(F) := J_{\underline{s}}(F) \cap J^K(F)$ and $I_{\underline{s}}(F) := J_{\underline{s}}(F) \cap I(F)$. For $e \in \{0, \infty\}^{\mathbb{N}_0}$ and $K > 0$, we define the sets

$$J_e(F) := \{z \in J(F) \ : \ \operatorname{addr}_F(z) \in \boldsymbol{\Sigma}_e\} = \bigcup_{\underline{s} \in \boldsymbol{\Sigma}_e} J_{\underline{s}}(F),$$

$J^K_e(F) := J_e(F) \cap J^K(F)$ and $I_e(F) := J_e(F) \cap I(F)$. If F is normalised, then $I_e(F) = J(F) \cap \exp^{-1}\left(I^{0,0}_e(f)\right)$.

There is a natural way to order the tracts with respect to the vertical position that they are attached to infinity. Using this, we can endow the set of sequences $\boldsymbol{\Sigma}_e$ with the lexicographic order.

Definition 3.45 (Lexicographic order). Let $F : \mathcal{T} \to H$ be a map in the class $\mathcal{B}^*_{\log}$. If T, T' are components of $\mathcal{T}_\infty$, then we say that $T < T'$ if T' is in the *upper* connected component of the intersection of a right half-plane and the complement of T. If T, T' are components of $\mathcal{T}_0$, then we say that $T < T'$ if T' is in the *lower* connected component of the intersection of a left half-plane and the complement of T. Finally, if $\underline{s}, \underline{s}' \in \mathbf{\Sigma}_e$ for some $e \in \{0, \infty\}^{\mathbb{N}_0}$, then we say that $\underline{s} < \underline{s}'$ if there is $k \in \mathbb{N}_0$ such that $T_n = T_n'$ for all $n < k$ and $T_k < T_k'$.

The set $\mathbf{\Sigma}_e$ endowed with the lexicographic order is a totally ordered space. Observe that, since the function F preserves orientation, if $T_1 < T_2$ in $\mathcal{T}_\infty$ and T is a component of $\mathcal{T}_0$, then with the lexicographic ordering we have $F_{|T}^{-1}(T_1) < F_{|T}^{-1}(T_2)$.

Sometimes it will be useful to consider a partition of the tracts of a function in the class $\mathcal{B}^*$ (and its logarithmic transforms) into *fundamental domains*. The following terminology was introduced by Rempe in [Rem08].

Definition 3.46 (Fundamental domain)**.** Let $f \in \mathcal{B}^*$ and let $F : \mathcal{T} \to H$ be a logarithmic transform of f that is in the class $\mathcal{B}^*_{log}$. Let $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ be the curve joining zero to infinity from Theorem 3.20.

(i) The set $\exp^{-1} \delta$ defines infinitely many *fundamental strips* S_n, $n \in \mathbb{Z}$. Every tract of F is contained in a fundamental strip.

(ii) For each tract T_n of F, the restriction $F_{|T_n} : T_n \to H$ is a one-to-one covering of either H_0 or H_∞. Hence, the set $F_{|T_n}^{-1}(H \setminus \exp^{-1} \delta)$ has infinitely many components $F_{n,i} \subseteq T_n$, $i \in \mathbb{Z}$, that we call *fundamental domains* of F.

(iii) Similarly, the preimages $f^{-1}(\delta)$ divide each tract V_n of f into infinitely many sets $D_{n,i} = \exp F_{m,i} \subseteq V_n$, $i \in \mathbb{Z}$, for some $m \in \mathbb{Z}$, that we call *fundamental domains* of f.

Note that sometimes we will refer to a sequence of fundamental domains using only one subindex when we do not need to specify whether two fundamental domains are a subset of the same tract or not.

Since the orbit of every point in $J(F)$ avoids $\exp^{-1}(\delta)$, we can define external addresses in terms of fundamental domains rather than tracts. This is the approach followed, for example, by Benini and Fagella [BF15]. However, since the image of each fundamental domain is contained in a fundamental strip, the fundamental domain F_n is determined by the tract T_n that contains F_n and the fundamental strip containing the next tract T_{n+1}. Thus, considering external addresses

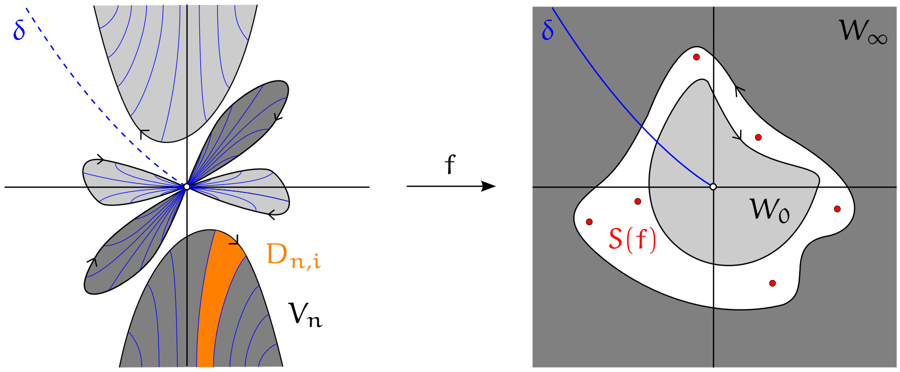


Figure 8: Fundamental domains of a function f in the class $\mathcal{B}^*$.

of fundamental domains does not add more information to the symbolic dynamics of F.

We can also consider external addresses for functions $f \in \mathcal{B}^*$ rather than for their logarithmic transforms. In this case, specifying the sequence of tracts in $\mathcal{V}$ does not capture the whole combinatorics of f; we define the external addresses of f in terms of fundamental domains. Let $\mathcal{A}_f$ denote the symbolic alphabet consisting of the fundamental domains of f.

Definition 3.47 (External address of f)**.** Let $f \in \mathcal{B}^*$ and let $F \in \mathcal{B}^*_{\log}$ be a *periodic* logarithmic transform of f. If $z = \exp w$, where $w \in J(F)$, we define the *external address* (under f) of z, $\operatorname{addr}_f(z)$, to be the symbol sequence $\underline{t} = (D_n) \in \mathcal{A}_f^{\mathbb{N}_0}$ such that $f^n(z) \in D_n$ for all $n \in \mathbb{N}_0$.

The next lemma describes the correspondence between external addresses of f and external addresses of a logarithmic transform F of f (see [BF15, Lemma 2.9]).

Lemma 3.48. *Let* $f \in \mathcal{B}^*$ *and let* $F \in \mathcal{B}^*_{\log}$ *be a logarithmic transform of* f*. If* $z = \exp w$*, then the external address* $\operatorname{addr}_f(z) = (D_n)$ *is uniquely determined by the external address* $\operatorname{addr}_F(w) = (T_n)$*. Conversely, if we have* $\operatorname{addr}_f(z) = (D_n)$*, then* $\operatorname{addr}_F(w) = (T_n)$ *is unique up to replacing* T_0 *by a* $2k\pi i$*-translate of* T_0 *for some* $k \in \mathbb{Z}$*.*

Proof. Let (T_n) be a sequence of tracts of F, then the sequence of fundamental domains $(D_n) \subseteq \mathcal{V}$ is given by $D_n = \exp F_n$ which, in turn, is determined by T_n and T_{n+1}.

On the other hand, if (D_n) is a sequence of fundamental domains of f, then the tract $T_0 \supseteq F_0$, where $\exp F_0 = D_0$, is given by the choice of the logarithmic transform F, which is unique up to addition of integer multiples of $2\pi i$, and the rest of tracts in the sequence (T_n) are determined by the fact that T_n is the only tract in the fundamental strip $F(F_{n-1})$ containing a component of $\exp^{-1}(D_n)$. ■

We say that a sequence of fundamental domains (D_n) of f is *admissible* if it corresponds to an admissible external address $\underline{s} \in \boldsymbol{\Sigma}$. In this chapter we use external addresses in terms of tracts mostly and restrict the use of fundamental domains to the times when we need them, in order to keep the notation simple.

3.6 UNBOUNDED CONTINUA IN THE JULIA SET

A priori, the set $J_{\underline{s}}(F)$ may be empty for some external addresses in $\underline{s} \in \boldsymbol{\Sigma}$. Recall that Rippon and Stallard [RS05b] showed that, for a general transcendental entire function f, the components of the fast escaping set $A(f) \subseteq I(f)$ are all unbounded. Using similar ideas, Rempe showed that if $f \in \mathcal{B}$ (and the same argument works for class $\mathcal{B}_{log}$), then every tract T contains an unbounded closed connected set A consisting of points that escape within T [Rem08, Theorem 2.4]. Sometimes we refer to an unbounded closed connected set $X \subseteq \mathbb{C}$ as an *unbounded continuum*; note, however, that such set is not a continuum in $\mathbb{C}$ as it is not compact, but $X \cup \{\infty\}$ is a continuum in $\hat{\mathbb{C}}$ (see Lemma 3.50).

Although [Rem08, Theorem 2.4] only concerns points that escape within a tract, if $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ is a periodic external address, then it follows that $J_{\underline{s}}(F)$ contains an unbounded continuum of escaping points. Indeed, if $\underline{s} = \overline{T_0 T_1 \dots T_{p-1}}$ has period $p \in \mathbb{N}$ and T_k, $0 \leqslant k < p$, are tracts of F, then there is a tract T of F^p contained in T_0 such that $F^k(T) \subseteq T_k$, $1 \leqslant k < p$, and the result follows from applying [Rem08, Theorem 2.4] to F^p in T.

It was remarked in [BJR12, p. 2107] that if $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ contains only finitely many symbols, then [Rem08, Theorem 2.4] can be adapted to show that $J_{\underline{s}}(F) \neq \emptyset$ and hence $J_{\underline{s}}(F)$ contains an unbounded continuum A; see [BF15, Proposition 2.11] for a detailed proof of this result. In [Rem07], Rempe showed that the set A can be chosen to be forward invariant. Later on, [BRS08, Theorem 1.1] generalised the result of

Rempe for transcendental meromorphic functions in $\mathbb{C}$ with tracts (not necessarily in the class $\mathcal{B}$).

For transcendental self-maps of $\mathbb{C}^*$, the components of the fast escaping set are unbounded in $\mathbb{C}^*$ (see Theorem 2.9). We recall that a set $X \subseteq \mathbb{C}^*$ is *unbounded* if its closure $\widehat{X}$ in $\hat{\mathbb{C}}$ contains zero or infinity. The following lemma is a combination of Theorems 2.4 and 2.9 and follows from the constructions in their proofs. Recall that $I_e^{0,0}(f) \subseteq I_e(f)$ is the set of escaping points whose essential itinerary is exactly e.

Lemma 3.49. *Let f be a transcendental self-map of $\mathbb{C}^*$. For each essential itinerary $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$, there exists an unbounded closed connected set $A_e \subseteq I_e^{0,0}(f)$ which consists of fast escaping points and whose closure $\widehat{A}_e$ in $\hat{\mathbb{C}}$ contains zero or infinity depending on the value of e_0.*

Lemma 3.49 implies that the set $J_e(F)$ contains at least one unbounded component. The goal of this section is to show that, under certain hypotheses, the set $J_{\underline{s}}(F)$ contains an unbounded continuum. We begin by stating the boundary bumping theorem [Nad92, Theorem 5.6] (see also [RRRS11, Theorem A.4]) which implies that if $X \subseteq \hat{\mathbb{C}}$ is a compact connected set containing zero or infinity and $E = X \cap \mathbb{C}^*$, then every component of E is unbounded in $\mathbb{C}^*$.

Lemma 3.50 (Boundary bumping theorem)**.** *Let X be a non-empty compact connected metric space and let $E \subsetneq X$ be non-empty. If C is a connected component of E, then $\partial C \cap \partial E \neq \emptyset$ (where boundaries are taken relative to X).*

First we show that if $J_{\underline{s}}^K(F) \neq \emptyset$ for sufficiently large $K > 0$, then the set $J_{\underline{s}}(F)$ contains an unbounded continuum. The following proposition is the analogue of [RRRS11, Lemma 3.3] for the class $\mathcal{B}^*_{\log}$. We include the proof for completeness.

Proposition 3.51. *Let $F \in \mathcal{B}^*_{\log}$. There exists $K_1(F) \geqslant 0$ such that if $K \geqslant K_1(F)$, for every $\underline{s} \in \boldsymbol{\Sigma}$, if $z_0 \in J_{\underline{s}}^K(F)$, then there exists an unbounded closed connected set $A \subseteq J_{\underline{s}}(F)$ with* $\operatorname{dist}(z_0, A) \leqslant 2\pi$.

Proof. We may assume without loss of generality that F is normalised with $H = \mathbb{H}_R^{\pm}$ for some $R > 0$. Let $K_1(F) > 0$ be large enough that if $K \geqslant K_1(F)$, then all bounded components of $H \cap \overline{\mathcal{T}}$ are in the vertical band $V_K := \{z \in \mathbb{C} : |\operatorname{Re} z| < K\}$. Note that the set V_K can only intersect a finite number of tracts in each fundamental strip.

Let $Y \subseteq \mathbb{C}$ be an unbounded continuum such that $Y \setminus B(F^k(z_0), 2\pi)$ has exactly one unbounded component. In that case we denote this component by $X_k(Y)$. Let $\underline{s} = (T_n) \in \boldsymbol{\Sigma}$. For all $k \geqslant 1$, we have that $\emptyset \neq X_k(\overline{T_k}) \subseteq H$ and hence $F^{-1}_{|T_{k-1}}$ maps $X_k(\overline{T_k})$ into T_{k-1}. By the expansivity property (3.3), since $\operatorname{dist}\big(F^k(z_0), X_k(T_k)\big) = 2\pi$, we have that $\operatorname{dist}\big(F^{k-1}(z_0), F^{-1}_{T_{k-1}}\big(X_k(T_k)\big)\big) \leqslant \pi$ and $X_{k-1}\big(F^{-1}_{T_{k-1}}\big(X_k(T_k)\big)\big) \neq \emptyset$. Thus we can define the sets

$$A_k := X_0\big(F^{-1}_{T_0}\big(\cdots\big(X_{k-1}\big(F^{-1}_{T_{k-1}}\big(X_k(T_k)\big)\big)\big)\cdots\big)\big) \quad \text{for } k \geqslant 1,$$

and we put $A_0 := X_0(\overline{T_0})$. Observe that here we are using the fact that $\underline{s} \in \boldsymbol{\Sigma}$ because $F^{-1}_{T_k}$ is only defined in one of the two components of H.

Let $\widehat{A}_k$ denote the closure of A_k in $\hat{\mathbb{C}}$ which is a continuum. By construction, $\widehat{A}_{k+1} \subseteq \widehat{A}_k$ and $\operatorname{dist}(z_0, A_k) \leqslant \pi$, thus

$$A' := \bigcap_{k \geqslant 0} \widehat{A}_k$$

is a continuum in $\hat{\mathbb{C}}$ and $A' \setminus \{0, \infty\}$ has a component A such that $\operatorname{dist}(z_0, A) \leqslant 2\pi$. Finally, by Lemma 3.50, the set A is unbounded in $\mathbb{C}^*$. ■

Next we show that, as in the entire case, if an external address $\underline{s} \in \boldsymbol{\Sigma}$ has only finitely many symbols, then the set $J_{\underline{s}}(F)$ contains an unbounded continuum. Note that in contrast to the previous proposition, now we need to show that $J_{\underline{s}}(F) \neq \emptyset$. We use the following lemma which is the analogue of [BF15, Proposition 2.6] for the class $\mathcal{B}^*$.

Lemma 3.52. *Let $F \in \mathcal{B}^*_{\log}$ have good geometry and let $\mathcal{F}$ be a finite union of fundamental domains of F. Then for any $K > 0$ sufficiently large,*

$$F^{-1}\big(\{z \in \mathbb{C} \,:\, |\operatorname{Re} z| = K\}\big) \cap \mathcal{F} \subseteq \{z \in \mathbb{C} \,:\, |\operatorname{Re} z| < K\}.$$

In the following proposition we adapt the proof of [BF15, Proposition 2.11] to our setting. This is based on the ideas of [Rem08, Theorem 2.4] and will be used later to prove Theorem 3.7.

Proposition 3.53. *Let $F \in \mathcal{B}^*_{\log}$. There exists $K_2(f) > 0$ such that if $K \geqslant K_2(F)$ and $\underline{s} \in \boldsymbol{\Sigma}$ contains finitely many different symbols, then the set $J^K_{\underline{s}}(F)$ contains a continuum whose points have unbounded real part.*

Proof. Suppose that $\underline{s} = (T_n)$ contains N different symbols for tracts $T_1^s, \ldots, T_N^s$ from $\mathcal{T}$, $N \in \mathbb{N}$, and choose fundamental domains $F_{j,k}^s \subseteq T_j^s$, $1 \leqslant j \leqslant N$, so that $F(F_{j,k}^s) \supseteq T_k^s$. Let $\mathcal{F}$ denote the finite collection of fundamental domains $\{F_{j,k}^s\}_{1\leqslant j \leqslant N}$ and assume that $K_2 = K_2(F) > 0$ is sufficiently large that Lemma 3.52 holds for $\mathcal{F}$ and $K > K_2(F)$. Then define (F_n) to be the sequence of fundamental domains from $\mathcal{F}$ satisfying $F_n \subseteq T_n$ and T_{n+1} lies in $F(F_n)$.

Let X_0 be the unbounded component of $F_0 \cap \mathbb{H}_K^{\pm}$ and, for each $n > 0$, let X_n be the unique unbounded component of

$$F_{|F_0}^{-1}(\cdots(F_{|F_{n-2}}^{-1}(F_{|F_{n-1}}^{-1}(F_n) \cap \mathbb{H}_K^{\pm}) \cap \mathbb{H}_K^{\pm}) \cdots) \cap \mathbb{H}_K^{\pm},$$

where $F_{|F_n}^{-1}$ is the branch of F^{-1} that maps the fundamental strip $F(F_n) \subseteq H$ in which F_{n+1} lies to the fundamental domain $F_n \subseteq T_n$. Note that since F is entire, $F_{|F_n}^{-1}$ maps unbounded sets to unbounded sets.

Lemma 3.52 tells us that $F^{-1}(\partial\mathbb{H}_K^{\pm}) \cap \mathcal{F} \subseteq \mathbb{C} \setminus \mathbb{H}_K^{\pm}$ and therefore for each $F_n \in \mathcal{F}$, necessarily $F_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$. Furthermore, if Y is an unbounded continuum with $Y \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$, then, by Lemma 3.52, $F_{|F_n}^{-1}(Y) \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$. Thus, since $F_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$, we have that $X_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$ for all $n \in \mathbb{N}_0$.

As before, let $\widehat{X}_n$ be the closures of X_n in $\hat{\mathbb{C}}$ and define

$$X' := \bigcap_{k \in \mathbb{N}_0} \widehat{X}_n$$

which is an unbounded continuum. Since all the unbounded continua $\widehat{X}_n$ intersect $\partial\mathbb{H}$, $X' \setminus \{0, \infty\}$ has a component X that intersects $\partial\mathbb{H}_K^{\pm}$ and is unbounded by Lemma 3.50. ∎

In particular, Proposition 3.53 covers all the periodic external addresses in $\mathbf{\Sigma}$. Observe that by considering external addresses that consist of fundamental domains instead of tracts we would obtain the result that for all such sequences containing only finitely many different fundamental domains of f there is an unbounded continuum consisting of escaping points whose orbit lies in that sequence of fundamental domains.

3.7 PROPERTIES OF DYNAMIC RAYS

In Theorem 3.1 we showed that bounded-type functions have no escaping Fatou components. Instead, escaping points often lie in curves tending to the essential singularities called *dynamic rays* or, sometimes, *hairs* such that in every unbounded proper subset of them, a *ray tail*, points escape uniformly. We say that a dynamic ray is *broken* if one of its forward iterates contains a critical point; this concept was introduced in [BF15, Definition 2.2].

Definition 3.54 (Dynamic ray)**.** Let f be a transcendental self-map of $\mathbb{C}^*$. A *ray tail* of f is an injective curve

$$\gamma : [0, +\infty) \to I(f)$$

such that $f^n(\gamma(t)) \to \{0, \infty\}$ as $t \to +\infty$ for all $n \in \mathbb{N}_0$ and also $f^n(\gamma(t)) \to \{0, \infty\}$ uniformly in t as $n \to \infty$. A *dynamic ray* of f is a maximal injective curve

$$\gamma : (0, +\infty) \to I(f)$$

such that $\gamma|_{[t,+\infty)}$ is a ray tail for every $t > 0$. Similarly, we can define ray tails for any logarithmic transform F of f, which is only defined on the set $\mathcal{T}$, and dynamic rays for any lift $\tilde{f}$ of f. We shall abuse the notation and use γ for both the ray as a set and its parametrization.

We say that a dynamic ray γ is *broken* if $f^n(\gamma)$ contains a critical point for $n \in \mathbb{N}_0$. A non-broken ray γ is said to *land* if $\overline{\gamma} \setminus \gamma$ consists of a single point or, in other words, if $\gamma(t)$ has a limit as $t \to 0$. We say that a dynamic ray γ is *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(\gamma) = \gamma$. If $f(\gamma) = \gamma$, then we say that γ is an *invariant* dynamic ray.

Example 3.55. We give a couple of straightforward examples of dynamic rays in $\mathbb{C}^*$.

(i) The positive real line is an invariant dynamic ray for the function $f(z) = \exp(z + 1/z)$, and points escape to infinity under iteration. This is an example of a broken ray because the function f has a critical point at $z = 1$.

(ii) If we now consider the function $g(z) = \exp(-z + 1/z)$, the positive real line is again forward invariant but $z = 1$ is a repelling fixed point of g. In this case, the intervals $(0,1)$ and $(1,+\infty)$ form a cycle of 2-periodic non-broken dynamic rays.

Observe that dynamic rays can land at an essential singularity and the limits of $\gamma(t)$ as $t \to 0$ and $t \to +\infty$ may even coincide. The dynamic ray from the following example is non-broken and goes from zero to infinity.

Example 3.56. The positive real line is an invariant non-broken dynamic ray for the function $f(z) = z\exp\big(z^2 + \exp(-1/z^2)\big)$ (see Figure 9).

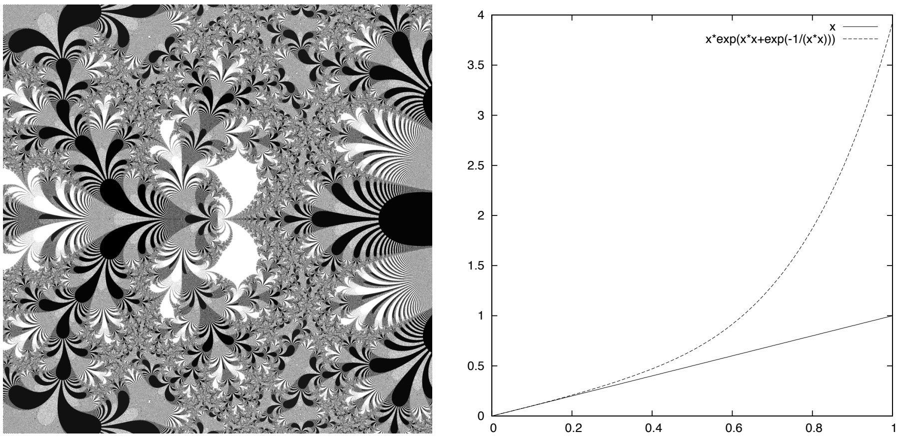


Figure 9: On the left, phase space of the function $f(z) = z\exp(z^2 + \exp(-1/z^2))$ from Example 3.56. On the right, the graph of the restriction of this function to the positive real line.

Since the exponential function is a local homeomorphism, we have the following correspondence between dynamic rays of transcendental self-maps of $\mathbb{C}^*$ and those of their lifts.

Lemma 3.57. *Let* f *be a transcendental self-map of* $\mathbb{C}^*$ *and let* $\tilde{f}$ *be a lift of* f*. Then* γ *is a dynamic ray of* f *if and only if any connected component* $\tilde{\gamma}$ *of* $\exp^{-1}\gamma$ *is a dynamic ray of* $\tilde{f}$*. Furthermore,* γ *lands or is broken if and only if* $\tilde{\gamma}$ *lands or is broken, respectively.*

It is a well-known result for entire functions that if the postsingular set is bounded then all periodic dynamic rays land. This was first proved for the exponential family [SZ03b; Rem06]. Rempe proved a more general version of the result for Riemann surfaces that applies to maps in the classes $\mathcal{B}$ and $\mathcal{B}^*$ [Rem08, Theorem B.1]; see also [Den14, Theorem 1.1] for an alternative proof of this result for the class $\mathcal{B}$. The same techniques imply the following result in our setting.

Proposition 3.58. *Let $f \in \mathcal{B}^*$ with postsingular set $P(f)$ bounded away from zero and infinity. Then all periodic dynamic rays of f land, and the landing points are either repelling or parabolic periodic points of f.*

Next we show that, since points in ray tails escape uniformly, each dynamic ray is contained in a set $I_e(f)$ for some essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$.

Lemma 3.59. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let γ be a dynamic ray of f. Then, for every ray tail $\gamma' \subseteq \gamma$, there is $\ell \in \mathbb{N}_0$ such that all the points in $f^\ell(\gamma')$ have the same essential itinerary. Hence, there exists an essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$ such that $\gamma \subseteq I_e(f)$.*

Proof. By definition, ray tails escape uniformly and hence, if γ' is a ray tail, there is $\ell \in \mathbb{N}$ such that $f^n(\gamma') \cap S^1 = \emptyset$ for all $n \geqslant \ell$. Then, all points in $f^\ell(\gamma')$ have the same essential itinerary; that is, in the notation of Section 2.1, $\gamma' \subseteq I_e^{\ell,0}(f)$ for some $e \in \{0, \infty\}^{\mathbb{N}_0}$.

Now suppose that γ is a dynamic ray with $z_1 \in \gamma \cap I_{e_1}(f)$ and $z_2 \in \gamma \cap I_{e_2}(f)$. Then there is a ray tail $\gamma' \supseteq \{z_1, z_2\}$ and $\ell \in \mathbb{N}$ such that all points in $f^\ell(\gamma')$ have the same essential itinerary. Thus, $e_1 \cong e_2$ and $\gamma \subseteq I_{e_1}(f) = I_{e_2}(f)$. ■

Actually, since all the images of a dynamic ray are unbounded in $\mathbb{C}^*$, dynamic rays are asymptotically contained in tracts which are preimages of the neighbourhood W of the set $\{0, \infty\}$. Furthermore, each dynamic ray is asymptotically contained in exactly one of the fundamental domains of the function F.

In the following proposition we show that, in order to prove Theorem 3.4, we only require that every escaping point has an iterate that is on a ray tail (see [RRRS11, Proposition 2.3]).

Proposition 3.60. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let $z_0 \in I(f)$. Suppose that some iterate $f^k(z_0)$, $k \in \mathbb{N}_0$, is on a ray tail γ_k of f. Then either z_0 is on a ray tail, or there is some $n \leqslant k$ such that $f^n(z_0)$ belongs to a ray tail that contains an asymptotic value of f.*

Proof. Suppose that $\gamma_k : [0, \infty) \to \mathbb{C}^*$ is a parametrization of such a ray tail and $\gamma_k(0) = f^k(z_0)$. Let $\gamma_{k-1} : [0, T) \to \mathbb{C}^*$ be a maximal lift of γ_k such that $\gamma_{k-1}(0) = f^{k-1}(z_0)$ and $f(\gamma_{k-1}(t)) = \gamma_k(t)$. If $T = \infty$, then $\gamma_{k-1}(t)$ must tend to zero or infinity as $t \to +\infty$, otherwise we would have $\gamma_{k-1}(t) \to a \in \mathbb{C}^*$ as $t \to +\infty$, so

$$f(a) = f\left(\lim_{t\to+\infty} \gamma_{k-1}(t)\right) = \lim_{t\to+\infty} f\left(\gamma_{k-1}(t)\right) = \lim_{t\to+\infty} \gamma_k(t) \in \{0, \infty\}$$

which is a contradiction. Thus, $f^{(k-1)}(z_0)$ is on a ray tail. Now consider the case that $T < \infty$ and let

$$w := \lim_{t\to T} \gamma_{k-1}(t) \in \hat{\mathbb{C}}.$$

Again, it cannot happen that $f(w) \in \{0, \infty\}$ because $\gamma_k(T)$ would be an asymptotic value, so $f(w) = \gamma_k(t_0)$ for some $t_0 \in [0, \infty)$. In this case, γ_{k-1} could be extended, contradicting its maximality. Note that if w was a critical point we would need to choose a branch of f^{-1}. Thus, $w \in \{0, \infty\}$ and $\gamma_k(T)$ is an asymptotic value of f (possibly zero or infinity). Then either we have a ray tail $\gamma_{k-1} \subseteq f^{-1}(\gamma_k) \subseteq I(f)$ connecting $f^{(k-1)}(z)$ to one of the essential singularities or γ_k contains an asymptotic value. The result follows from applying the above reasoning inductively. ■

Note that Proposition 3.60 can also be proved by applying its version for entire functions to a lift $\tilde{f}$ of f and then use the correspondence from Lemma 3.57.

We conclude this section by stating a result about escaping points that follows from the expansivity property (3.3) in Lemma 3.21 (see [RRRS11, Lemma 3.2] for the analogous result for entire functions).

Lemma 3.61. *Let $F : \mathcal{T} \to H$ be in the class $\mathcal{B}^{*n}_{log}$ with $H = \mathbb{H}^{\pm}_R$ for some $R > 0$. If $z, w \in J_{\underline{s}}(F)$ for some external address $\underline{s}$ and $z \neq w$, then*

$$\lim_{k\to+\infty} \max\{|\operatorname{Re} F^k(z)|, |\operatorname{Re} F^k(w)|\} = +\infty. \tag{3.7}$$

Observe that (3.7) does not imply that neither the point z nor w escape because both points may have an unbounded orbit but with a subsequence where their iterates are bounded. In the next section we will introduce a condition for F (see Definition 3.62) which implies that, in the situation of Lemma 3.61, both points z and w escape, and hence all points in $J_{\underline{s}}(F)$ except possibly one must escape.

Lemma 3.24, Lemma 3.61 and Proposition 3.51 correspond, respectively, to Lemma 3.1, Lemma 3.2 and Theorem 3.3 in [RRRS11, Section 3] and constitute the main tools to prove Theorem 3.4 in the next section.

3.8 ESCAPING POINTS AND DYNAMIC RAYS

In this section we adapt the results in [RRRS11, Sections 4 and 5] to our setting. Since the proof Theorem 3.4 follows closely that of [RRRS11, Theorem 1.2], we only sketch it and emphasize the differences between them.

The head-start condition is designed so that every escaping point is mapped eventually to a ray tail and hence we are able to apply Proposition 3.60 and conclude that either the point itself is in a ray tail or some iterate is in a ray tail that contains a singular value.

Definition 3.62 (Head-start condition)**.** Let $F : \mathcal{T} \to H$ be a function in the class $\mathcal{B}^*_{log}$. We first define the *head-start condition* for tracts, then for external addresses and finally for logarithmic transforms.

- Let T, T' be two tracts in $\mathcal{T}$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a (not necessarily strictly) monotonically increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}_+$. We say that the pair (T, T') satisfies the *head-start condition* for φ if, for all $z, w \in \overline{T}$ with $F(z), F(w) \in \overline{T'}$,

$$|\text{Re } w| > \varphi(|\text{Re } z|) \Rightarrow |\text{Re } F(w)| > \varphi(|\text{Re } F(z)|).$$

- We say that an external address $\underline{s} = (T_n) \in \mathbf{\Sigma}$ satisfies the *head-start condition* for φ if all consecutive pairs of tracts (T_n, T_{n+1}) satisfy the head-start condition for φ, and if for all distinct $z, w \in J_{\underline{s}}(F)$, there is $M \in \mathbb{N}_0$ such that either $|\text{Re } F^M(z)| > \varphi(|\text{Re } F^M(w)|)$ or $|\text{Re } F^M(w)| > \varphi(|\text{Re } F^M(z)|)$.

- We say that F satisfies a *head-start condition* if every external address of F satisfies the head-start condition for some φ. If the same function φ can be chosen for all external addresses, we say that F satisfies the uniform head-start condition for φ.

Notice that in the second part we require that the head-start condition cannot be a void condition for any itinerary. Furthermore, if

$|\mathrm{Re}\ F^M(z)| > \varphi(|\mathrm{Re}\ F^M(w)|)$ and the head-start condition is satisfied for all consecutive pairs of tracts (T_n, T_{n+1}) for $n \geqslant M$, then we have $|\mathrm{Re}\ F^n(z)| > \varphi(|\mathrm{Re}\ F^n(w)|)$ for all $n > M$.

The head-start condition allows us to order the points in $J_{\underline{s}}(F)$ by the growth of the absolute value of their real parts.

Definition 3.63 (Speed ordering)**.** Let $\underline{s} \in \mathbf{\Sigma}$ be an external address satisfying the head-start condition for a function φ. For $z, w \in J_{\underline{s}}(F)$, we say that $z \succ w$ if there exists $K \in \mathbb{N}_0$ such that $|\mathrm{Re}\ F^K(z)| > \varphi(|\mathrm{Re}\ F^K(w)|)$. We extend this order to the closure $\widehat{J_{\underline{s}}}(F)$ in $\hat{\mathbb{C}}$ by the convention that $0, \infty \succ z$ for all $z \in J_{\underline{s}}(F)$.

Note that although a dynamic ray may contain both zero and infinity in its closure in $\hat{\mathbb{C}}$, ray tails are a subset of $\mathcal{T}$ and hence their closure contains either zero or infinity.

The head-start condition implies that the speed ordering is a total order on the set $\widehat{J_{\underline{s}}}(F)$: if there were two values $M_1, M_2 \in \mathbb{N}_0$ such that $|\mathrm{Re}\ F^{M_1}(z)| > \varphi(|\mathrm{Re}\ F^{M_1}(w)|)$ and $|\mathrm{Re}\ F^{M_2}(w)| > \varphi(|\mathrm{Re}\ F^{M_2}(z)|)$ then we would get a contradiction because once we are in one of these situations and the head-start condition is satisfied then it is preserved by iteration, that is, for example, if $|\mathrm{Re}\ F^{M_1}(z)| > \varphi(|\mathrm{Re}\ F^{M_1}(w)|)$, then $|\mathrm{Re}\ F^n(z)| > \varphi(|\mathrm{Re}\ F^n(w)|)$ for all $n > M_1$. Therefore $z \succ w$ if and only if there exists $n_0 \in \mathbb{N}_0$ such that $|\mathrm{Re}\ F^n(z)| > |\mathrm{Re}\ F^n(w)|$ for all $n > n_0$, and hence the speed ordering does not depend on the choice of the function φ.

Lemma 3.64. *Let $\underline{s} \in \mathbf{\Sigma}_e$, $e \in \{0, \infty\}^{\mathbb{N}_0}$, be an external address that satisfies the head-start condition for a function φ. Then the order topology induced by the speed ordering $\succ$ on $\widehat{J_{\underline{s}}}(F)$ coincides with its topology as a subset of $\hat{\mathbb{C}}$ and, in particular, every connected component of $\widehat{J_{\underline{s}}}(F)$ is an arc.*

Moreover, there exists $K' > 0$ independent of $\underline{s}$ such that $J_{\underline{s}}^{K'}(F)$ is either empty or contained in the unique unbounded component of $J_{\underline{s}}(F)$, which is an arc to the essential singularity e_0 all of whose points escape except possibly its finite endpoint.

Proof. The first part follows from the fact that the identity map $\mathrm{id} : \widehat{J_{\underline{s}}}(F) \to (\widehat{J_{\underline{s}}}(F), \prec)$ is a homeomorphism (see [RRRS11, Theorem 4.4]). Indeed, for all $a \in J_{\underline{s}}(F)$, the sets

$$(a, +\infty)_{\prec} := \{z \in \widehat{J_{\underline{s}}}(F) \ : \ a \prec z\}, \quad (-\infty, a)_{\prec} := \{z \in \widehat{J_{\underline{s}}}(F) \ : \ z \prec a\},$$

are open sets in $\widehat{J}_{\underline{s}}(F)$ with the subspace topology of $\hat{\mathbb{C}}$: let $k \in \mathbb{N}_0$ be minimal with the property that $|\text{Re } F^k(a)| > \varphi(|\text{Re } F^k(z)|)$ then, by continuity, this inequality holds in a neighbourhood of z. Since $\widehat{J}_{\underline{s}}(F)$ with the order topology is Hausdorff, the map id^{-1} is continuous as well. The theorem follows from the order characterisation of the arc (see [RRRS11, Theorem A5]).

For the second part, if $K \geqslant K_1(F)$, where $K_1(F) \geqslant 0$ is the constant from Proposition 3.51, and $J^K_{\underline{s}}(F) \neq \emptyset$, then $J^K_{\underline{s}}(F)$ has an unbounded component A which is an arc to infinity. Since e_0 is the largest element of $\widehat{J}_{\underline{s}}(F)$ in the speed ordering, the set $\widehat{J}_{\underline{s}}(F)$ has only one unbounded component. Using the head-start condition, it can be shown that if $z, w \in J_{\underline{s}}(F)$ and $w \succ z$ then $w \in I_{\underline{s}}(F)$ (see [RRRS11, Corollary 4.5]). Finally, the fact that $J^{K'}_{\underline{s}}(F) \subseteq A$ for some $K' > K$ follows from the expansivity of F (see [RRRS11, Proposition 4.6]). ∎

As in the entire case, the following theorem can be deduced from Lemma 3.64 (see [RRRS11, Theorem 4.2]).

Theorem 3.65. *Let $F \in \mathcal{B}^*_{\log}$ satisfy a head-start condition. Then, for every escaping point z, there exists $k \in \mathbb{N}_0$ such that $F^k(z)$ is on a ray tail γ. This ray tail is the unique arc in $J(F)$ connecting $F^k(z)$ to either zero or infinity (up to reparametrization).*

Observe that Theorem 3.65 together with Proposition 3.60 imply that if f is a transcendental self-map of $\mathbb{C}^*$ and $z \in I(f)$, then either z is on a ray tail or there is some $n \leqslant k$ such that $f^n(z)$ belongs to a ray tail that contains an asymptotic value of f.

Previously we have seen that if f has finite order then any logarithmic transforms F of f has good geometry in the sense of Definition 3.28. To complete the proof of Theorem 3.4 we show that functions with good geometry satisfy a head-start condition.

Theorem 3.66. *Let $F \in \mathcal{B}^{*n}_{\log}$ be a function with good geometry. Then the function F satisfies a linear head-start condition.*

Proof. Let $\underline{s} \in \mathbf{\Sigma}$ be an external address and suppose that F has bounded slope with constants (α, β). Then the orbits of any two points $z, w \in J_{\underline{s}}(F)$ eventually separate far enough one from the other.

More precisely, if $K \geqslant 1$, there exist a constant $\delta = \delta(\alpha, \beta, K) > 0$ such that if $|z - w| \geqslant \delta$, then either

$$|\operatorname{Re} F^n(z)| > K|\operatorname{Re} F^n(w)| + |z - w|$$

or the same condition, exchanging the roles of z and w, holds for all $n \geqslant 1$ (see [RRRS11, Lemma 5.2]). Hence the external address $\underline{s}$ satisfies the second part of the head-start condition with the linear function $\varphi(x) = Kx + \delta$.

It remains to check that if $\underline{s} = (T_n)$, then for all $k \in \mathbb{N}_0$ and for all $z, w \in T_k$ such that $F(z), F(w) \in T_{k+1}$, we have

$$|\operatorname{Re} w| > K|\operatorname{Re} z| + \delta \quad \Rightarrow \quad |\operatorname{Re} F(w)| > K|\operatorname{Re} F(z)| + \delta.$$

We omit the technical computations from this proof, which are identical to the ones for the entire case, and just observe that this follows from the fact that the tracts of F have uniformly bounded wiggling with constants K and μ for some $\mu > 0$ if and only if the conditions

$$|\operatorname{Re} w| > K|\operatorname{Re} z| + M'$$

$$|\operatorname{Im} F(z) - \operatorname{Im} F(w)| \leqslant \alpha \max\{|\operatorname{Re} F(z)|, |\operatorname{Re} F(w)|\} + \beta$$

imply that $|\operatorname{Re} F(w)| > K|\operatorname{Re} F(z)| + M'$ whenever $z, w \in T$, for some $M' > 0$. Hence F satisfies the uniform linear head-start condition with constants K and M for some $M > 0$ (see [RRRS11, Proposition 5.4]). ■

Finally we prove Theorem 3.4 concerning the existence of dynamic rays for compositions of finite order transcendental self-maps of $\mathbb{C}^*$.

Proof of Theorem 3.4. Let $f_1, \dots, f_n$ be finite order transcendental self-maps of $\mathbb{C}^*$ for some $n \geqslant 1$. By Theorem 3.3, the functions f_i are in the class $\mathcal{B}^*$. Composing the functions f_i with affine changes of variable, we can assume that each f_i has a normalised logarithmic transform $F_i : \mathcal{T}_i \to \mathbb{H}^{\pm}_{R_i} \in \mathcal{B}^{*n}_{\log}$ for some $R_i > 0$.

By Proposition 3.36, each F_i has good geometry and hence, by Theorem 3.66, they all satisfy linear head-start conditions. Just as for functions in $\mathcal{B}_{\log}$, linear head-start conditions are preserved by composition in $\mathcal{B}^*_{\log}$ (see [RRRS11, Lemma 5.7]). If F_1 has bounded slope and all F_i satisfy uniform linear head-start conditions, then the function $F := F_n \circ \cdots \circ F_1 \in \mathcal{B}^*_{\log}$, which is a logarithmic transform of $f = f_n \circ \cdots \circ f_1 \in \mathcal{B}^*$, has bounded slope and satisfies a uniform linear head-start condition when restricted to a suitable set of tracts.

Finally, we can apply Theorem 3.65 and Proposition 3.60 to conclude that every point $z \in I(f)$ is on a ray tail that joins z to either zero or infinity. ■

Remark 3.67. The proof of Theorem 3.4 relies on normalised logarithmic transforms. However, it is possible to carry out the same ideas using only disjoint-type functions, so that the resulting function F is also of disjoint type (see [RRRS11, Theorem 5.10] and [Bar07, Theorem C]).

3.9 PERIODIC RAYS AND CANTOR BOUQUETS

In Section 3.6 we observed that the set $J_{\underline{s}}(F)$ may be empty for some $\underline{s} \in \mathbf{\Sigma}$. For transcendental entire functions in the exponential family, $f_\lambda(z) = \lambda e^z$, $\lambda \neq 0$, there is a characterization of which external addresses give rise to hairs, and this led to the notion of *exponentially bounded* (or *admissible*) external addresses in that context (see [SZ03a]). In particular, every periodic external address is exponentially bounded. Observe that the term admissible has a different meaning in this context.

Barański, Jarque and Rempe [BJR12] studied the set of dynamic rays for the functions considered in [RRRS11] and [Bar07], and showed that they have uncountably many rays organised in a Cantor bouquet (see Definition 3.68). In this section we adapt their techniques to study the set of dynamic rays constructed in Section 3.8.

We begin by proving Theorem 3.7, which states that if $f \in \mathcal{B}^*$ satisfies the hypothesis of Theorem 3.4 and $\underline{t} = (D_n)$ is an admissible external address of f which contains finitely many symbols, then f has a unique (non-empty) dynamic ray with that external address. Furthermore, if (D_n) is periodic and the postsingular set $P(f)$ is bounded, then the dynamic ray lands.

Proof of Theorem 3.7. By Proposition 3.53, there exists an unbounded continuum $A \subseteq \mathcal{V}$ of escaping points with external address $\underline{t} = (D_n)$.

Let F be a periodic logarithmic transform of f, and let $\underline{s} = (T_n)$ be the external address that corresponds to the sequence of fundamental domains (D_n) of f by Lemma 3.48. By Theorem 3.4, the set $J_{\underline{s}}(F)$ is a dynamic ray $\tilde{\gamma}$, and the projection $\gamma = \exp \tilde{\gamma}$ is a dynamic ray of f with external address $\underline{t} = (D_n)$. Finally, by Lemma 3.58, since $P(f)$ is bounded, all periodic rays land. ■

Theorem 3.7 implies, for example, that each fundamental domain D of f contains exactly one invariant ray because the constant external address $\underline{t} = (D_n)$ with $D_n = D$ for all $n \in \mathbb{N}_0$ is unique.

In Lemma 3.49, which summarized some results from Chapter 2, we saw that if f is any transcendental self-map of $\mathbb{C}^*$ and $e \in \{0, \infty\}^{\mathbb{N}_0}$, then the set $I_e^{0,0}(f)$ contains an unbounded closed connected subset A_e. Furthermore, if $f \in \mathcal{B}^*$ and satisfies the hypothesis of Theorem 3.4, then Theorem 3.7 implies that the set $I_e^{0,0}(f)$ contains a ray tail; note that a dynamic ray may intersect the unit circle and hence contain points that are not in $I_e^{0,0}(f)$. Therefore, in this case, since the set $\{0, \infty\}^{\mathbb{N}_0}$ has uncountably many non-equivalent sequences e and two such sequences give disjoint sets $I_e(f)$, the escaping set $I(f)$ contains uncountably many rays.

As stated in the introduction, a stronger result is true, namely Theorem 3.8, which states that for every essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$, the set $I_e^{0,0}(f)$ contains a *Cantor bouquet* and, in particular, uncountably many hairs. With the goal in mind of proving this theorem, we start by giving a precise definition of a Cantor bouquet (see [AO93, Definition 1.2]).

Definition 3.68 (Cantor bouquet)**.** A set $B \subseteq [0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

(a) The set B is a closed subset of $\mathbb{R}^2$.

(b) For every point $(x, y) \in B$, there exists a value $t_y \geqslant 0$ such that $\{x : (x, y) \in B\} = [t_y, +\infty)$.

(c) The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$, there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$ for all $n \in \mathbb{N}$, and $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

The set $[t_y, +\infty) \times \{y\}$ is called the *hair attached at* y and the point (t_y, y) is called its *endpoint*. A *Cantor bouquet* is a set $X \subseteq \mathbb{C}$ that is the image of a straight brush under a homeomorphism of $\mathbb{C}$ or $\mathbb{C}^*$.

First we are going to show that, for each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, the set $J(F)$ contains an *absorbing set* X_e consisting of hairs such that every point in the set $I_e(F)$ enters X_e after finitely many iterations (see [RRRS11, Theorem 4.7]). Recall that, for $e \in \{0, \infty\}^{\mathbb{N}_0}$, we defined the set

$$J_e(F) := \{z \in J(F) \; : \; \operatorname{addr}_F(z) \in \boldsymbol{\Sigma}_e\} = \bigcup_{\underline{s} \in \boldsymbol{\Sigma}_e} J_{\underline{s}}(F).$$

It will be helpful to use the following notation: for each $e \in \{0, \infty\}^{\mathbb{N}_0}$, we define the set of sequences

$$\boldsymbol{\Sigma}_e^+ := \bigcup_{n \in \mathbb{N}} \sigma^n(\boldsymbol{\Sigma}_e)$$

and the set

$$J_e^+(F) := \{z \in J(F) \; : \; \operatorname{addr}_F(z) \in \boldsymbol{\Sigma}_e^+\} = \bigcup_{n \in \mathbb{N}} J_{\sigma^n(e)}(F),$$

which is forward invariant.

Proposition 3.69. *Suppose that* $F \in \mathcal{B}_{\log}^*$ *satisfies a head-start condition. Then, for every* $e \in \{0, \infty\}^{\mathbb{N}_0}$*, there exists a closed subset* $X_e \subseteq J_e^+(F)$ *with the following properties:*

(a) $F(X_e) \subseteq X_e$.

(b) The connected components of X_e *are closed arcs to infinity all of whose points except possibly its endpoint escape.*

(c) Every point in $I_e(F)$ *enters the set* X_e *after finitely many iterations.*

If the function F *is of disjoint type, then we may choose* $X_e = J_e^+(F)$ *and if* F *is* $2\pi i$*-periodic, then* X_e *can also be chosen to be* $2\pi i$*-periodic.*

Proof. Let X'_e be the union of all unbounded components of the set $J_e(F)$, and define the set

$$X_e := \bigcup_{n \in \mathbb{N}} X'_{\sigma^n(e)}.$$

Since unbounded components of $J(F)$ map to unbounded components of $J(F)$ by F, we have $F(X'_e) \subseteq X'_{\sigma(e)}$ and hence X_e is forward invariant.

By Lemma 3.50, the closure $\widehat{X}_e$ in $\hat{\mathbb{C}}$ is the connected component of $J_e^+(F) \cup \{\infty\}$ that contains infinity and hence the set X_e is closed. By Lemma 3.64, the set X_e consists of arcs to infinity all of whose points except possibly its endpoint escape.

Let $K' > 0$ be the constant from Lemma 3.64, independent of $\underline{s} \in \mathbf{\Sigma}$, so that $J_{\underline{s}}^{K'}(F)$ is either empty or contained in the unbounded component of $J_{\underline{s}}(F)$, which is contained in X_e if $\underline{s} \in \mathbf{\Sigma}_e^+$. Then (c) follows from the fact that points in $I_e(F)$ enter a set $J_{\sigma^n(e)}^{K'}(F) \subseteq X_e$, $n \in \mathbb{N}$, after finitely many iterations.

Finally, recall from Definition 3.18 that functions in the class $\mathcal{B}^*_{\log}$ are of the form $F : \mathcal{T} \to H_0 \cup H_\infty$, where the sets H_0 and H_∞ contain, respectively, a left and a right half-plane. If F is of disjoint type, then

$$J_e(F) \cup \{\infty\} = \bigcup_{\underline{s} \in \mathbf{\Sigma}_e} \bigcap_{n \in \mathbb{N}} \left(F^{-1}_{|T_0}\left(\cdots F^{-1}_{|T_{n-2}}\left(F^{-1}_{|T_{n-1}}(\overline{H}_{e_n})\right)\cdots\right) \cup \{\infty\}\right),$$

which is a union of nested intersections of unbounded continua, hence every component of $J_e(F)$ is an unbounded continuum and we can choose $X_e = J_e(F)$. If F is a $2\pi i$-periodic function, then the set X'_e is also $2\pi i$-periodic. ■

Following [BJR12], the strategy to prove Theorem 3.8 will be, for each essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$, to compactify the space of admissible external addresses $\mathbf{\Sigma}_e$ by adding a *circle of addresses at infinity* to show that the set X'_e (and hence X_e) contains a Cantor bouquet.

Lemma 3.70. *For every $e \in \{0, \infty\}^{\mathbb{N}_0}$, there exists a totally ordered set $\tilde{\mathcal{S}}_e \supseteq \mathbf{\Sigma}_e$, where the order on $\tilde{\mathcal{S}}_e$ agrees with the lexicographic order on $\mathbf{\Sigma}_e$, such that*

(a) with the order topology, the set $\tilde{\mathcal{S}}_e$ is homeomorphic to $\mathbb{R} \cup \{-\infty, +\infty\}$;

(b) the set $\mathbf{\Sigma}_e$ is dense in $\tilde{\mathcal{S}}_e$.

The construction of $\tilde{\mathcal{S}}_e$ is achieved by defining *intermediate entries* of each set $\mathcal{T}_{e_0}^{e_1}$ with $e_0, e_1 \in \{0, \infty\}$, that is, symbols which correspond to entries in between pairs of adjacent tracts as well as to limits of sequences of tracts. We then add *intermediate external addresses* to the set $\mathbf{\Sigma}_e$, that is, finite sequences of the form $\underline{s} = T_0 T_1 \ldots T_{n-1} S_n$, where

$T_j \in \mathcal{T}_{e_j}^{e_{j+1}}$, $0 \leqslant j < n$, and S_n is an intermediate entry of the set $\mathcal{T}_{e_n}^{e_{n+1}}$. We refer to [BJR12, Section 5] for the details.

We can then define a topology on the set $\tilde{H}_e := \overline{H}_{e_0} \cup \tilde{\mathcal{S}_e}$ that agrees with the induced topology on H and such that $\tilde{H}_e$ is homeomorphic to the closed unit disc. Then, in this topology, the closure $\tilde{X}_e$ of the set X_e from Proposition 3.69 is a *comb*, a compactification of a straight brush, with the arc $\tilde{\mathcal{S}}_e$ as base.

Definition 3.71 (Comb)**.** A *comb* is a continuum X containing an arc B, called the *base* of the comb, such that

(a) the closure of every component of $X \setminus B$ is an arc with exactly one endpoint in the base B;

(b) the intersection of the closures of any two hairs is empty;

(c) the set $X \setminus B$ is dense in X.

The fact that a Cantor bouquet consists of uncountably many hairs comes from the fact that a perfect set is uncountable. We introduce now the concept of (one-sided) hairy arc, a comb where every hair is accumulated by other hairs.

Definition 3.72 (Hairy arc)**.** A *hairy arc* is a comb with base B and an order $\prec$ on B such that if $b \in B$ and x belongs to the hair attached at b, then there exist sequences (x_n^+) and (x_n^-), attached respectively at points $b_n^+, b_n^- \in B$, such that $b_n^- \prec b \prec b_n^+$ and $x_n^-, x_n^+ \to x$ as $n \to \infty$. A *one-sided hairy arc* is a hairy arc with all its hairs attached to the same side of the base.

Given a straight brush, it is easy to see that we can add a base to obtain a hairy arc. Aarts and Oversteegen showed that one-sided hairy arcs (and, in particular, straight brushes) are all *ambiently homeomorphic* to each other, that is, they can be mapped to each other by a homeomorphism of $\mathbb{C}$, and hence the converse of the previous statement is also true [AO93, Theorem 4.1].

Lemma 3.73. *Let X be a one-sided hairy arc with base B. Then there is a homeomorphism of $\mathbb{C}$ that maps $X \setminus B$ to a straight brush.*

In order to show that X_e contains a Cantor bouquet, we prove that every hair in X'_e is accumulated by hairs of the same set from both sides. To do so, we adapt the proof of [BJR12, Proposition 7.3].

Proposition 3.74. *Let $F : \mathcal{T} \to H$ be a $2\pi i$-periodic function in the class $\mathcal{B}^*_{\log}$, and let $\underline{e} \in \{0, \infty\}^{\mathbb{N}_0}$ and $\tau > 0$. Then there exists $\tau' \geqslant \tau$ such that for every $z_0 \in J^{\tau'}_{\underline{e}}(F)$, there exist sequences $(z^-_n), (z^+_n) \subseteq J^{\tau}_{\underline{e}}(F)$ with addresses $\operatorname{addr}(z^-_n) < \operatorname{addr}(z_0) < \operatorname{addr}(z^+_n)$ for all $n \in \mathbb{N}$ and $z^-_n, z^+_n \to z_0$ as $n \to \infty$.*

Proof. Let R_0 be the constant from Lemma 3.21 so that $\mathbb{H}^{\pm}_R \subseteq H$ and $|F'(z)| \geqslant 2$ for $|\operatorname{Re} z| \geqslant R_0$. Let $n \in \mathbb{N}$, and let $\varphi_n : H_{e_n} \to H_{e_0}$ be the branch of F^{-n} that maps $F^n(z_0)$ to z_0. Set $\tau' := \max\{R, \tau\} + \pi$ and, for $n \in \mathbb{N}$, define

$$z^{\pm}_n := \varphi_n\big(F^n(z_0) \pm 2\pi i\big) \in J^{\tau}_{\underline{e}}(F).$$

Then $\operatorname{addr}(z^-_n) < \operatorname{addr}(z_0) < \operatorname{addr}(z^+_n)$ for all n. Finally, since F is expanding with respect to the Euclidean metric on $\mathbb{H}^{\pm}_R$, the maps φ_n are contractions and $z^{\pm}_n \to z_0$ as $n \to \infty$. ∎

Note that given any logarithmic transform F of a function $f \in \mathcal{B}^*$ we can modify it to obtain a periodic logarithmic transform $\hat{F}$ of f by adding a suitable multiple of $2\pi i$ to F on each of its tracts.

Finally we sketch the proof of Theorem 3.8. The main idea is to use the existence of a potential function ρ that 'straightens' the brush $X'_{\underline{e}}$ (see [BJR12, Proposition 7.1]).

Proof of Theorem 3.8. Let $F \in \mathcal{B}^*_{\log}$ be $2\pi i$-periodic and satisfy a uniform head-start condition and let $X'_{\underline{e}}$ denote the union of the unbounded components of $J_{\underline{e}}(F)$ as in Proposition 3.69. For each sequence $\underline{e} \in \{0, \infty\}^{\mathbb{N}_0}$, consider the set

$$Z_{\underline{e}} := \{z \in X'_{\underline{e}} \;:\; \rho\big(F^j(z)\big) \geqslant K \text{ for all } j \in \mathbb{N}_0\} \cup \tilde{S}_{\underline{e}},$$

where ρ is a $2\pi i$-periodic continuous function that is strictly increasing on the hairs and such that $\rho(z_n) \to +\infty$ if and only if $|\operatorname{Re} z_n| \to +\infty$. Then, there exists $R > 0$ sufficiently large so that

$$J^R_{\underline{e}}(F) \subseteq Z_{\underline{e}} \subseteq \tilde{X}_{\underline{e}}$$

and hence $Z_{\underline{e}}$ is a comb. Then Proposition 3.74 together with the fact that F satisfies a uniform head-start condition imply that $Z_{\underline{e}}$ is a hairy arc and, by Lemma 3.73, there is a homeomorphism from $\hat{\mathbb{C}} \setminus \{e_0\}$ to $\mathbb{C}$ that maps $Z_{\underline{e}} \setminus \tilde{S}_{\underline{e}}$ to a straight brush. We can choose the set $X_{\underline{e}}$ from Proposition 3.69 to be $2\pi i$-periodic and so both $J_{\underline{e}}(F)$

and $\exp(J_e(F))$ contain an absorbing Cantor bouquet. Note that all the points in $\exp(J_e(F))$ belong to $I_e^{0,0}(f)$ except, possibly, the finite endpoints of the hairs.

Finally, if F is of disjoint type, then the closure of $J_e(F)$ in $\tilde{H}_e$ is a one-sided hairy arc, and hence both $J_e(F)$ and $\exp(J_e(F))$ are Cantor bouquets. ■

4

ESCAPING FATOU COMPONENTS

In this chapter we use approximation theory to construct examples of transcendental self-maps of $\mathbb{C}^*$ with escaping wandering domains and Baker domains that accumulate at $\{0, \infty\}$ in any possible way and also give the first explicit examples, in closed form, of transcendental self-maps of $\mathbb{C}^*$ that have escaping Fatou components. Our results provide completely new examples of transcendental entire functions with escaping Fatou components.

4.1 INTRODUCTION AND MAIN RESULTS

Here we are concerned with escaping points in the Fatou set. By normality, if a Fatou component U contains a point in $I(f)$, then $U \subseteq I(f)$. Moreover, any two points in an escaping Fatou component U have, eventually, the same essential itinerary and hence we can associate an essential itinerary to U which is unique up to equivalence. If f is a transcendental self-map of $\mathbb{C}^*$, we have proved that $I_e(f) \cap J(f) \neq \emptyset$ for each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$ (see Theorem 2.4). Therefore it is a natural question whether for each $e \in \{0, \infty\}^{\mathbb{N}_0}$ we can find a transcendental self-map of $\mathbb{C}^*$ with a Fatou component in $I_e(f)$.

In the Introduction, we mentioned that several people used approximation theory to provide examples of transcendental self-maps of $\mathbb{C}^*$ with escaping wandering domains. However, in our notation, all the previous examples had essential itinerary $e \in \{\overline{\infty}, \overline{0}, \overline{\infty 0}\}$. The following result provides examples of transcendental self-maps of $\mathbb{C}^*$ that have an escaping wandering domain with any prescribed essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$. In particular, we obtain wandering domains whose essential itinerary is not periodic.

Theorem 4.1. *For each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map f of $\mathbb{C}^*$ such that $\operatorname{ind}(f) = n$ and $I_e(f)$ contains a wandering domain.*

A result of Cowen [Cow81] on holomorphic self-maps of $\mathbb{D}$ whose Denjoy-Wolff point lies on $\partial\mathbb{D}$ led to the following classification of Baker domains by Fagella and Henriksen [FH06], where U/f is the Riemann surface obtained by identifying points of U that belong to the same orbit under f:

- a Baker domain U is *hyperbolic* if U/f is conformally equivalent to $\{z \in \mathbb{C} : -s < \operatorname{Im} z < s\}/\mathbb{Z}$ for some $s > 0$;
- a Baker domain U is *simply parabolic* if U/f is conformally equivalent to $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}/\mathbb{Z}$;
- a Baker domain U is *doubly parabolic* if U/f is conformally equivalent to $\mathbb{C}/\mathbb{Z}$.

Note that this classification does not require f to be entire and is valid also for Baker domains of transcendental self-maps of $\mathbb{C}^*$. König [Kön99] provided a geometric characterisation for each of these types (see Lemma 4.11). It is known that if U is a doubly parabolic Baker domain, then $f_{|U}$ is not univalent, but if U is a hyperbolic or simply parabolic Baker domain, then $f_{|U}$ can be either univalent or multivalent. Several examples of each type had been constructed, and recently Bergweiler and Zheng completed the table of examples by constructing a transcendental entire function with a simply parabolic Baker domain in which the function is not univalent [BZ12, Theorem 1.1].

The only previous examples of Baker domains of transcendental self-maps of $\mathbb{C}^*$ that the author is aware of are due to Kotus [Kot90], where she used approximation theory to construct two functions with invariant hyperbolic Baker domains escaping to zero and to infinity respectively. The following theorem provides examples of functions with Baker domains that have *any* periodic essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$.

Theorem 4.2. *For each periodic sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map f of $\mathbb{C}^*$ such that $\operatorname{ind}(f) = n$ and $I_e(f)$ contains a hyperbolic Baker domain.*

We also give the first explicit examples of transcendental self-maps of $\mathbb{C}^*$ with wandering domains and Baker domains. They all have the property that in a neighbourhood of infinity they behave like known examples of transcendental entire functions with wandering domains and Baker domains; see Section 2 for the details.

Example 4.3. The following transcendental self-maps of $\mathbb{C}^*$ have escaping Fatou components:

(i) The function $f(z) = z\exp\left(\frac{\sin z}{z} + \frac{2\pi}{z}\right)$ has a bounded wandering domain in which points escape to infinity (see Example 4.4).

(ii) The function $f(z) = 2z\exp\big(\exp(-z) + 1/z\big)$ has a hyperbolic Baker domain in which points escape to infinity that contains a right half-plane (see Example 4.12).

(iii) The function $f(z) = z\exp\left((e^{-z}+1)/z\right)$ has a doubly parabolic Baker domain in which points escape to infinity that contains a right half-plane (see Example 4.13).

It seems hard to find explicit examples of functions with Baker domains and wandering domains with any given essential itinerary, but it would be interesting to have an explicit example of a function with an escaping Fatou component that accumulates to both zero and infinity. It also seems difficult to find explicit examples of functions with simply parabolic Baker domains.

If f is a transcendental self-map of $\mathbb{C}^*$ with a wandering domain, then any lift $\tilde{f}$ of f has a wandering domain, while if f has a Baker domain, then $\tilde{f}$ has either a Baker domain (of the same type) or a wandering domain (see Lemmas 4.7 and 4.14).

Finally, observe that our constructions using approximation theory also provide new examples of transcendental entire functions with no zeros in $\mathbb{C}^*$ that have wandering domains and Baker domains.

Structure of the chapter. In Sections 4.2 and 4.3 we prove that the functions from Example 4.3 have the escaping Fatou components that we state. In Section 4.4 we introduce the tools from approximation theory that we will use in the proof of Theorem 4.1 in Section 4.5, and Theorem 4.2 in Section 4.6, to construct transcendental self-maps of $\mathbb{C}^*$ with escaping wandering domains and Baker domains, respectively. In Section 4.6 we also construct transcendental entire and meromorphic functions that are self-maps of $\mathbb{C}^*$ and have Baker domains.

4.2 EXPLICIT FUNCTIONS WITH WANDERING DOMAINS

As mentioned before, the author is not aware of any previous explicit examples of transcendental self-maps of $\mathbb{C}^*$ with wandering domains or Baker domains as all such functions were constructed using approximation theory.

Kotus [Kot90] showed that transcendental self-maps of $\mathbb{C}^*$ can have escaping wandering domains by constructing examples of such functions using approximation theory. Here we give an explicit example of such a function by modifying a transcendental entire function that has a wandering domain.

Example 4.4. The function $f(z) = z\exp\left(\frac{\sin z}{z} + \frac{2\pi}{z}\right)$ is a transcendental self-map of $\mathbb{C}^*$ which has a bounded wandering domain that escapes to infinity (see Figure 10).

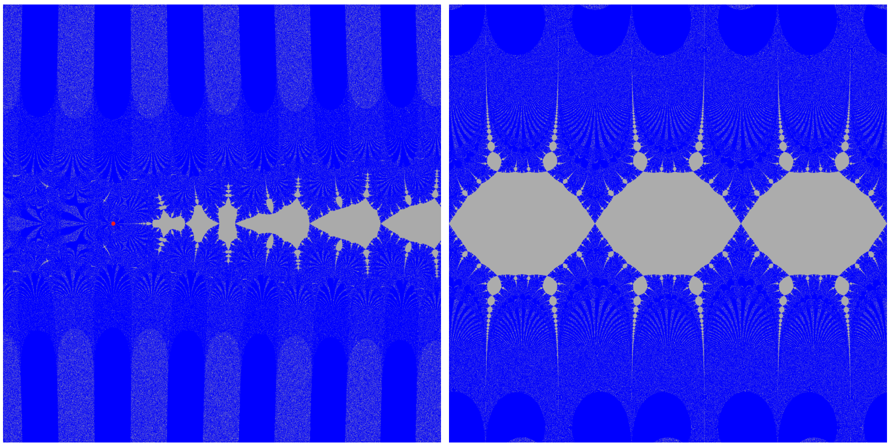


Figure 10: Phase space of the function $f(z) = z\exp\left(\frac{\sin z}{z} + \frac{2\pi}{z}\right)$ from Example 4.4 which has a wandering domain. On the right, the wandering domain for large values of $\operatorname{Re} z$.

Baker [Bak84, Example 5.3] (see also [RS08, Example 2]) studied the dynamics of the transcendental entire function $f_1(z) = z + \sin z + 2\pi$ that has a wandering domain containing the point $z = \pi$ that escapes to infinity. Observe that the function f from Example 4.4 satisfies that

$$f(z) = z + \sin z + 2\pi + o(1) \qquad \text{as } \operatorname{Re} z \to +\infty \tag{4.1}$$

in a horizontal band defined by $|\operatorname{Im} z| < K$ for some $K > 0$.

We first prove a general result which gives a sufficient condition that implies that a function has a bounded wandering domain (see

Figure 11) using some of the ideas from [RS08, Lemma 7(c)]. Given a doubly connected open set A, we define the inner boundary, $\partial_{in}A$, and the outer boundary, $\partial_{out}A$, of A to be the boundary of the bounded and unbounded complementary components of A respectively.

Lemma 4.5. *Let f be a function that is holomorphic on $\mathbb{C}^*$, let M be an affine map, let A be a doubly connected closed set in $\mathbb{C}^*$ with bounded complementary component B, and let $C \subseteq B$ be compact. Put*

$$A_n := M^n(A), \quad B_n := M^n(B) \quad \text{and} \quad C_n := M^n(C) \quad \text{for } n \in \mathbb{N}_0,$$

and suppose that

- $A_n \cup B_n \subseteq \mathbb{C}^*$ *for* $n \in \mathbb{N}_0$,
- *the sets* $\{B_n\}_{n\in\mathbb{N}_0}$ *are pairwise disjoint,*
- $f(\partial_{in} A_n) \subseteq C_{n+1}$ *for* $n \in \mathbb{N}_0$,
- $f(\partial_{out} A_n) \subseteq (A_{n+1} \cup B_{n+1})^c$ *for* $n \in \mathbb{N}_0$.

Then f has wandering domains $\{U_n\}_{n\in\mathbb{N}_0}$ such that

$$\partial_{in} A_n \subseteq U_n \quad \text{and} \quad \partial U_n \subseteq A_n \quad \text{for } n \in \mathbb{N}_0.$$

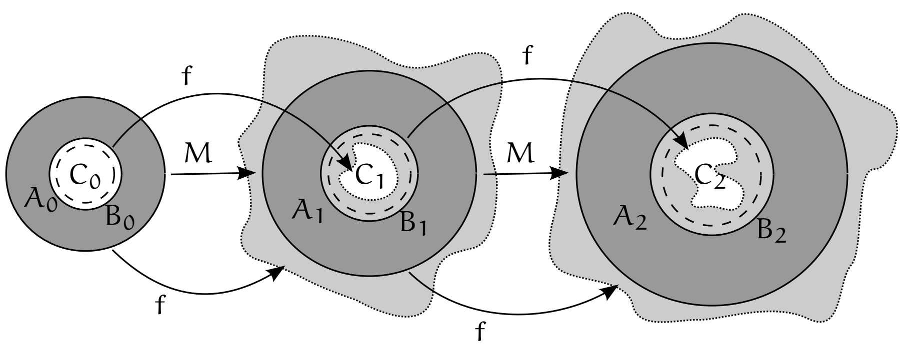


Figure 11: Sketch of the construction in Lemma 4.5.

In order to prove this lemma, we first need the following result on limit functions of holomorphic iterated function systems by Keen and Lakic [KL03, Theorem 1].

Lemma 4.6. *Let X be a subdomain of the unit disc $\mathbb{D}$. Then all limit functions of any sequence of functions (F_n) of the form*

$$F_n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1 \quad \text{for } n \in \mathbb{N},$$

where $f_n : \mathbb{D} \to X$ is a holomorphic function for all $n \in \mathbb{N}$, are constant functions in $\overline{X}$ if and only if $X \neq \mathbb{D}$.

We now proceed to prove Lemma 4.5.

Proof of Lemma 4.5. Since $f(\overline{B_n}) \subseteq C_{n+1} \subseteq B_{n+1}$, the iterates of f on each set $\overline{B_n}$ omit more than three points and hence, by Montel's theorem, the sets $\{\overline{B_n}\}_{n \in \mathbb{N}_0}$ are all contained in $F(f)$. For $n \in \mathbb{N}_0$, let U_n denote the Fatou component of f that contains $\overline{B_n}$. We now show that the functions

$$\Phi_k(z) := M^{-k}(f^k(z)) \quad \text{for } k \in \mathbb{N}_0,$$

form a normal family in U_n for all $n \in \mathbb{N}_0$.

Suppose first that the Fatou components $\{U_n\}_{n \in \mathbb{N}_0}$ are not distinct. Then there are two sets B_m and B_{m+p} with $m \in \mathbb{N}_0$ and $p > 0$ which lie in the same Fatou components $U_m = U_{m+p}$. Then, since $f^p(B_m) \subseteq B_{m+p}$ and $B_n \to \infty$ as $n \to \infty$, U_m must be periodic and in $I(f)$, and hence a Baker domain.

Let $z_m \in B_m$ and let K be any compact connected subset of U_m such that $K \supseteq B_m$. Then by Baker's distortion lemma (see Lemma 2.22), there exist constants $C(K) > 1$ and $n_0 \in \mathbb{N}_0$ such that

$$|f^k(z)| \leqslant C(K)|f^k(z_m)| \quad \text{for } z \in K,\ k \geqslant n_0.$$

Since M, and hence M^{-k}, is an affine transformation, M^{-k} preserves the ratios of distances, so

$$|\Phi_k(z)| = |M^{-k}(f^k(z))| \leqslant C(K)|M^{-k}(f^k(z_m))| = C(K)|z'_m|$$

where $z'_m \in B_m$ satisfies $M^k(z'_m) = f^k(z_m)$. Hence the family $\{\Phi_k\}_{k \in \mathbb{N}_0}$ is locally uniformly bounded on U_m, and hence is normal on U_m.

Suppose next that the Fatou components $\{U_n\}_{n \in \mathbb{N}_0}$ are disjoint. In this case we consider the sequence of functions

$$\varphi_k(z) := M^{-(k+1)}(f(M^k(z))) \quad \text{for } k \in \mathbb{N}_0,$$

which are defined on U_n, for $n \in \mathbb{N}_0$. Then

$$\Phi_k(z) = (\varphi_{k-1} \circ \cdots \circ \varphi_1 \circ \varphi_0)(z) = M^{-k}(f^k(z)) \quad \text{for } k \in \mathbb{N}_0. \tag{4.2}$$

Since the Fatou components $\{U_n\}_{n \in \mathbb{N}_0}$ are pairwise disjoint and

$$f^k(U_n) \subseteq U_{n+k},$$

we deduce that

$$f^k(U_n) \cap B_{n+k+1} = \emptyset$$

and hence

$$\Phi_k(U_n) \cap B_{n+1} = \emptyset \quad \text{for } k, n \in \mathbb{N}_0.$$

Thus $\{\Phi_k\}_{k \in \mathbb{N}_0}$ is normal on each U_n, by Montel's theorem, as required.

Now take $n \in \mathbb{N}_0$, and let $\{\Phi_{k_j}\}_{j \in \mathbb{N}_0}$ be a locally uniformly convergent subsequence of $\{\Phi_k\}_{k \in \mathbb{N}_0}$ on B_n. Note that

$$M^k(B_n) = B_{n+k} \quad \text{so} \quad f(M^k(B_n)) \subseteq C_{n+k+1}$$

and hence, for $k \in \mathbb{N}_0$,

$$\varphi_k(B_n) = M^{-(k+1)}(f(M^k(B_n))) \subseteq M^{-(k+1)}(C_{n+k+1}) = C_n.$$

We can now apply Lemma 4.6, after a Riemann mapping from B_n to the open unit disc $\mathbb{D}$, to deduce from (4.2) that there exists $\alpha_n \in \overline{B_n}$ such that, for all $z \in U_n$,

$$\Phi_{k_j}(z) \to \alpha_n \quad \text{as } j \to \infty.$$

To complete the proof that U_n is bounded by $\partial_{\text{out}} A_n$ for all $n \in \mathbb{N}$, suppose to the contrary that there is a point $z_0 \in \partial_{\text{out}} A_n$ that lies in U_n for some $n \in \mathbb{N}$. Let $\gamma \subseteq U_n$ be a curve that joins z_0 to a point $z_1 \in B_n$. Since γ is compact, $\Phi_{k_j}(\gamma) \to \alpha$ as $j \to \infty$ which contradicts the fact that $f^k(\gamma) \cap \partial_{\text{out}} A_{n+k} \neq \emptyset$ for all $k \in \mathbb{N}$ (this follows from the hypothesis that $f(\partial_{\text{out}} A_n) \subseteq (A_{n+1} \cup B_{n+1})^c$ for $n \in \mathbb{N}_0$). Thus, $\partial U_n \subseteq A_n$ for all $n \in \mathbb{N}$, and so the proof is complete. ∎

We now use Lemma 4.5 to show that the function f from Example 4.4 has a bounded wandering domain that escapes to infinity along the positive real axis.

Proof of Example 4.4. The entire function $g(z) = z + \sin z$ has superattracting fixed points at the odd multiples of π. For $n \in \mathbb{N}_0$, take $B_n := D((2n+1)\pi, r)$ and $C_n := D((2n+1)\pi, r/2)$ for some $r > 0$ sufficiently small that $g(B_n) \subseteq C_n$ and put

$$R_n := \{z \in \mathbb{C} \ : \ |\operatorname{Re} z - (2n+1)\pi| \leqslant 3\pi/2, \ |\operatorname{Im} z| \leqslant 3\}.$$

It follows from a straightforward computation that $g(\partial R_n) \subseteq R_n^c$ for all $n \in \mathbb{N}_0$ (see Figure 12).

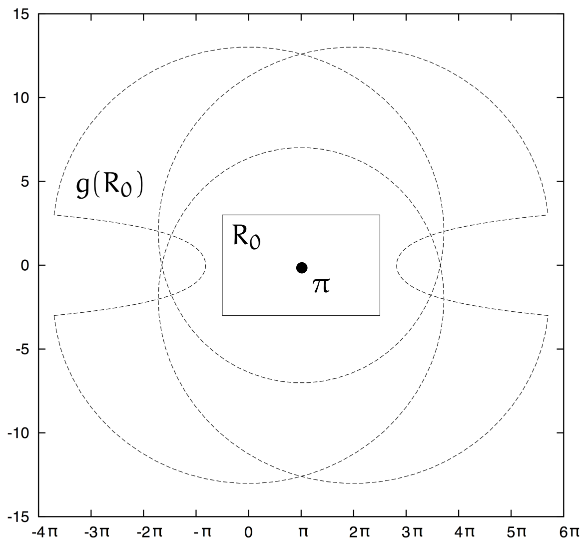


Figure 12: Rectangle R_0 and its image under $g(z) = z + \sin z$.

Then, by (4.1), there exists $N \in \mathbb{N}_0$ such that $f(B_n) \subseteq C_{n+1}$ and $f(\partial R_n) \subseteq R_{n+1}^c$ for all $n > N$. Thus, we can apply Lemma 4.5 to f with $M(z) = z + 2\pi$ and $A_n := R_n \setminus B_n$ for $n > N$ and conclude that the function f has wandering domains U_n that contain B_n and whose boundary is contained in R_n. ∎

The next lemma relates the wandering domains of a transcendental self-map of $\mathbb{C}^*$ and a lift of it.

Lemma 4.7. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of f. Then, if U is a wandering domain of f, every component of $\exp^{-1}(U)$ is a wandering domain of $\tilde{f}$ which must be simply connected.*

Proof. By a result of Bergweiler [Ber95], every component of $\exp^{-1}(U)$ is a Fatou component of $\tilde{f}$. Let U_0 be a component of $\exp^{-1}(U)$ and suppose to the contrary that there exist $m, n \in \mathbb{N}_0$, $m \neq n$, and a point $z_0 \in \tilde{f}^m(U_0) \cap \tilde{f}^n(U_0)$. Then, there exists points $z_1, z_2 \in U_0$ such that

$$f^m(e^{z_1}) = \exp \tilde{f}^m(z_1) = \exp z_0 = \exp \tilde{f}^n(z_2) = f^n(e^{z_2}).$$

Since $e^{z_1}, e^{z_2} \in U$, this contradicts the assumption that U is a wandering domain of f. Hence U_0 is a wandering domain of $\tilde{f}$.

Finally, by [Bak87, Theorem 1], the Fatou component U is either simply connected or doubly connected and surrounds the origin. Since the exponential function is periodic, taking a suitable branch of the logarithm one can show that the components of $\exp^{-1}(U)$ are simply connected. ∎

Remark 4.8. Observe that the converse of Lemma 4.7 does not hold. If f is a transcendental self-map of $\mathbb{C}^*$ with an attracting fixed point z_0 and A is the immediate basin of attraction of z_0, then there is a lift $\tilde{f}$ of f such that a component of $\exp^{-1}(A)$ is a wandering domain.

If a transcendental self-map of $\mathbb{C}^*$ has an escaping wandering domain, then we can use the previous lemma to obtain automatically an example of a transcendental entire function with an escaping wandering domain.

Example 4.9. The transcendental entire function $\tilde{f}(z) = z + \frac{\sin e^z}{e^z} + \frac{2\pi}{e^z}$, which is a lift of the function f from Example 4.4, has infinitely many grand orbits of bounded wandering domains that escape to infinity.

4.3 EXPLICIT FUNCTIONS WITH BAKER DOMAINS

We now turn our attention to Baker domains. As we mentioned in the introduction, Baker domains can be classified into hyperbolic, simply parabolic and doubly parabolic according to the Riemann surface U/f obtained by identifying the points of the Baker domain U that belong to the same orbit under iteration by the function f. König [Kön99] introduced the following notation.

Definition 4.10 (Conformal conjugacy)**.** Let $U \subseteq \mathbb{C}$ be a domain and let $f : U \to U$ be analytic. Then a domain $V \subseteq U$ is *absorbing* (or *funda-*

mental) for f if V is simply connected, $f(V) \subseteq V$ and for each compact set $K \subseteq U$, there exists $N = N_K$ such that $f^N(K) \subseteq V$. Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The triple (V, ϕ, T) is called a *conformal conjugacy* (or *eventual conjugacy*) of f in U if

(a) V is absorbing for f;

(b) $\phi : U \to \Omega \in \{\mathbb{H}, \mathbb{C}\}$ is analytic and univalent in V;

(c) $T : \Omega \to \Omega$ is a bijection and $\phi(V)$ is absorbing for T;

(d) $\phi(f(z)) = T(\phi(z))$ for $z \in U$.

In this situation we write $f \sim T$.

Observe that properties (b) and (d) imply that f is univalent in V. König also provided the following geometrical characterization of the three types of Baker domains [Kön99, Theorem 3]. This characterisation is also valid for any simply connected Baker domain of a transcendental self-map of $\mathbb{C}^*$.

Lemma 4.11. *Let U be a p-periodic Baker domain of a meromorphic function f in which $f^{np} \to \infty$ and on which f^p has a conformal conjugacy. For $z_0 \in U$, put*

$$c_n = c_n(z_0) := \frac{|f^{(n+1)p}(z_0) - f^{np}(z_0)|}{\operatorname{dist}(f^{np}(z_0), \partial U)}.$$

Then exactly one of the following cases holds:

(a) U is hyperbolic and $f^p \sim T_1(z) := \lambda z$ with $\lambda > 1$, which is equivalent to

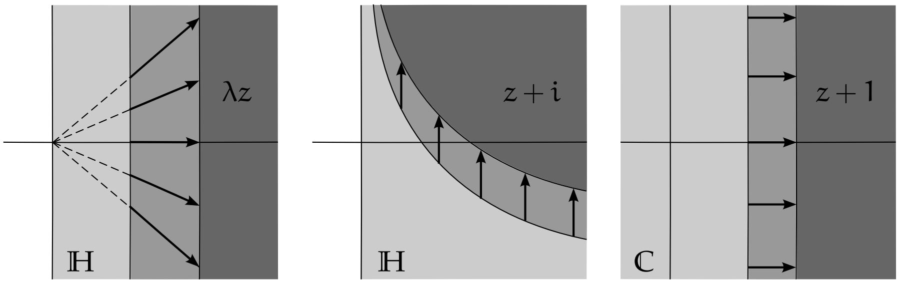
$$c_n > c \quad \textit{for } z_0 \in U,\ n \in \mathbb{N}, \quad \textit{where } c = c(f) > 0.$$

(b) U is simply parabolic and $f^p \sim T_2(z) := z \pm i$, which is equivalent to

$$\liminf_{n\to\infty} c_n > 0 \quad \textit{for } z_0 \in U, \quad \textit{but } \inf_{z_0 \in U} \limsup_{n\to\infty} c_n = 0;$$

(c) U is doubly parabolic and $f^p \sim T_3(z) := z + 1$, which is equivalent to

$$\lim_{n\to\infty} c_n = 0 \quad \textit{for } z_0 \in U.$$



(a) U hyperbolic (b) U simply parabolic (c) U doubly parabolic

Figure 13: Classification of Baker domains with their absorbing domains.

We now give a couple of explicit examples of transcendental self-maps of $\mathbb{C}^*$, with a hyperbolic and a doubly parabolic Baker domain, respectively.

Example 4.12. For every $\lambda > 1$, the function $f_\lambda(z) = \lambda z \exp(e^{-z} + 1/z)$ is a transcendental self-map of $\mathbb{C}^*$ which has an invariant, simply connected, hyperbolic Baker domain $U \subseteq \mathbb{C}^* \setminus \mathbb{R}_-$ whose boundary contains both zero and infinity, and the points in U escape to infinity (see Figure 14).

Proof of Example 4.12. First observe that

$$\begin{aligned} f_\lambda(z) &= \lambda z \exp\left(e^{-z} + \tfrac{1}{z}\right) \\ &= \lambda z \left(1 + e^{-z} + \tfrac{1}{2!}e^{-2z} + \cdots\right)\left(1 + \tfrac{1}{z} + \tfrac{1}{2!}\tfrac{1}{z^2} + \cdots\right) \\ &= \lambda z \left(1 + O\left(\tfrac{1}{z}\right)\right) \text{ as } \operatorname{Re} z \to \infty. \end{aligned} \tag{4.3}$$

Hence f_λ maps $\mathbb{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}$ into itself, for $R > 0$ sufficiently large, so $\mathbb{H}_R \subseteq U$, where U is an invariant Fatou component of f_λ. Also, for real $x > 0$,

$$f_\lambda(x) = \lambda x \exp\left(e^{-x} + \tfrac{1}{x}\right) > \lambda x > x$$

so $f_\lambda^n(x) \to \infty$ as $n \to \infty$. Hence, U is an invariant Baker domain of f which contains $(0, +\infty)$, so its boundary contains zero and infinity.

To show that U is a hyperbolic Baker domain, consider $z_0 \in U$. By the contraction property of the hyperbolic metric in U, the orbit of z_0 escapes to infinity in $\mathbb{H}_R$. Hence, by (4.3) and since $0 \in U^c$,

$$c_n = \frac{|f^{n+1}(z_0) - f^n(z_0)|}{\operatorname{dist}(f^n(z_0), \partial U)} \geqslant \frac{\lambda f^n(z_0)\left(1 + O\left(\frac{1}{f^n(z_0)}\right)\right) - f^n(z_0)}{|f^n(z_0)|}$$

$$> \lambda - 1 - \frac{O(1)}{|f^n(z_0)|} \text{ as } n \to \infty,$$

so

$$\liminf_{n\to\infty} c_n \geqslant \lambda - 1 > 0,$$

and hence U is hyperbolic.

Finally, observe that the negative real axis is invariant under f, so $(-\infty, 0) \cap U = \emptyset$ and hence U is simply connected. ∎

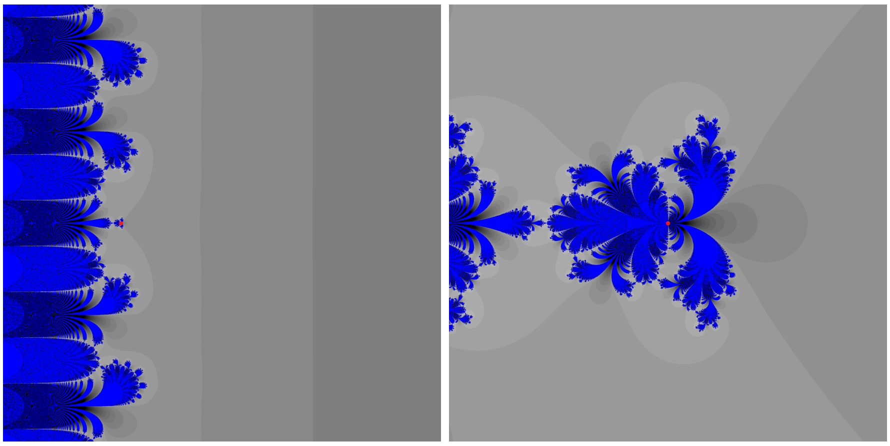


Figure 14: Phase space of the function $f_2(z) = 2z\exp(e^{-z} + 1/z)$ from Example 4.12. On the right, zoom of a neighbourhood of zero.

The function $f(z) = 2z\exp(e^{-z} + 1/z)$ has a repelling fixed point in the negative real line. If we choose $h(z) = 1/z^2$ instead of $1/z$, then $f(z) = 2z\exp(e^{-z} + 1/z^2)$ has the positive real axis in a Baker domain while the negative real axis is in the fast escaping set.

We now give a second explicit example of transcendental self-map of $\mathbb{C}^*$ with a Baker domain which, in this case, is doubly parabolic.

Example 4.13. The function $f(z) = z\exp((e^{-z} + 1)/z)$ is a transcendental self-map of $\mathbb{C}^*$ which has an invariant, simply connected, doubly parabolic Baker domain $U \subseteq \mathbb{C}^* \setminus \mathbb{R}_-$ whose boundary contains both zero and infinity, and the points in U escape to infinity (see Figure 15).

Proof of Example 4.13. Looking at the power series expansion of f, we have

$$\begin{aligned} f(z) &= z\exp\left(\frac{e^{-z}}{z}+\frac{1}{z}\right) \\ &= z\left(1+\frac{e^{-z}}{z}+\frac{1}{2!}\frac{e^{-2z}}{z^2}+\cdots\right)\left(1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\cdots\right) \\ &= z\left(1+\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right) \quad \text{as } \operatorname{Re} z \to \infty. \end{aligned}$$

Therefore f maps the right half-plane $\mathbb{H}_R := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > R\}$ into itself for sufficiently large values of $R > 0$ and $\mathbb{H}_R$ is contained in an invariant Baker domain U of f, in which $\operatorname{Re} f^n(z) \to +\infty$ as $n \to \infty$. Since $f(x) > x$ for all $x > 0$, the positive real axis lies in U. Let $z_0 \in U$, then

$$f^{n+1}(z_0)-f^n(z_0)=f^n(z_0)\left(1+O\left(\frac{1}{f^n(z_0)}\right)\right)-f^n(z_0)=O(1) \text{ as } n\to\infty$$

and, if R is as above,

$$\operatorname{dist}(f^n(z_0), \partial U) \geqslant \operatorname{Re} f^n(z_0) - R \quad \text{ as } n \to \infty,$$

so

$$c_n = \frac{|f^{n+1}(z_0)-f^n(z_0)|}{\operatorname{dist}(f^n(z_0), \partial U)} \leqslant \frac{O(1)}{\operatorname{Re} f^n(z_0) - R} \to 0 \quad \text{ as } n \to \infty.$$

Thus, by Lemma 4.11, the Baker domain U is doubly parabolic.

Finally, observe that, for $x \in (-\infty, 0)$, $f^n(x) \to \infty$ along the negative real axis as $n \to \infty$, so $(-\infty, 0) \cap U = \emptyset$ and hence U is simply connected. ∎

Lemma 4.14. *Let f be a transcendental self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of f. Then, if U is a Baker domain of f, every component U_k, $k \in \mathbb{Z}$, of $\exp^{-1}(U)$ is either a (preimage of a) Baker domain or a wandering domain of $\tilde{f}$. Moreover, if U is simply connected and U_k is a Baker domain, then U_k is hyperbolic, simply parabolic or doubly parabolic if and only if U is hyperbolic, simply parabolic or doubly parabolic, respectively.*

Proof. By [Ber95], every component of $\exp^{-1}(U)$ is a Fatou component of $\tilde{f}$. Moreover, since $\exp^{-1}(I(f)) \subseteq I(\tilde{f})$, U_k is either a Baker domain, a preimage of a Baker domain or an escaping wandering domain of $\tilde{f}$.

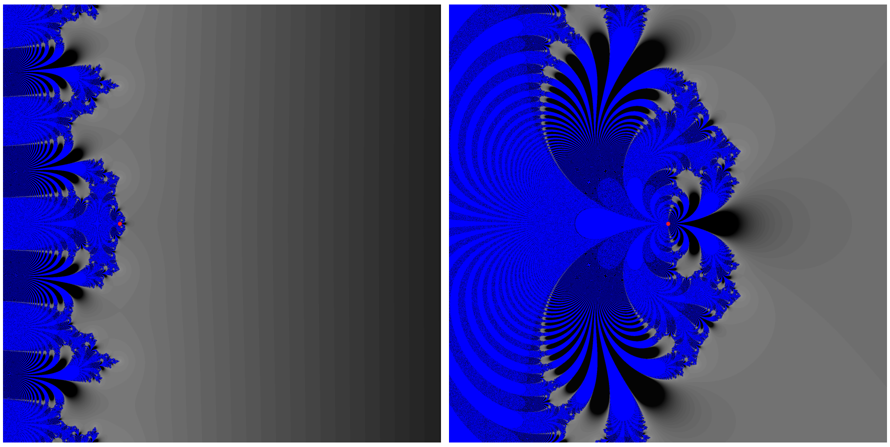


Figure 15: Phase space of the function $f(z) = z\exp((e^{-z}+1)/z)$ from Example 4.13. On the right, zoom of a neighbourhood of zero.

Suppose that U has period $p \geqslant 1$ and U_k is periodic. Then the Baker domain U_k has period q with $p \mid q$. Let (V, ϕ, T) be a conformal conjugacy of f^q in U. Then $(\tilde{V}, \tilde{\phi}, T)$ is a conformal conjugacy of $\tilde{f}^q$ in U_k, where $\tilde{V}$ is the component of $\exp^{-1} V$ that lies in U_k and $\tilde{\phi} = \phi \circ \exp$. Thus, the Baker domains U and U_k are of the same type. ∎

As before, we use Lemma 4.14 to provide examples of transcendental entire functions with Baker domains and wandering domains.

Example 4.15. The entire function $\tilde{f}(z) = \ln\lambda + z + \exp(-e^z) + e^{-z}$, which is a lift of the function f from Example 4.12, has an invariant hyperbolic Baker domain that contains the real line.

Example 4.16. The entire function $\tilde{f}(z) = z + \frac{\exp(-e^z)}{e^z} + e^{-z}$, which is a lift of the function f from Example 4.13, has an invariant doubly parabolic Baker domain that contains the real line.

4.4 PRELIMINARIES ON APPROXIMATION THEORY

In this section we state the results from approximation theory that will be used in Sections 4.5 and 4.6 to construct examples of functions with wandering domains and Baker domains, respectively. We follow the terminology from [Gai87, Chapter IV], and introduce Weierstrass and Carleman sets. Recall that if $F \subseteq \mathbb{C}$ is a closed set, then $A(F)$ denotes the set of continuous functions $f : F \to \mathbb{C}$ that are holomorphic in the interior of F.

Definition 4.17 (Weierstrass set)**.** We say that a closed set $F \subseteq \mathbb{C}$ is a *Weierstrass set* in $\mathbb{C}$ if each $f \in A(F)$ can be approximated by entire functions *uniformly* on F; that is, for every $\varepsilon > 0$, there is an entire function g for which

$$|f(z) - g(z)| < \varepsilon \quad \text{for all } z \in F.$$

The next result is due to Arakelyan and provides a characterisation of Weierstrass sets [Ara64]. In the case that $F \subseteq \mathbb{C}$ is compact and $\mathbb{C} \setminus F$ is connected, then it follows from Mergelyan's theorem [Gai87, Theorem 1 on p. 97] that functions in $A(F)$ can be uniformly approximated on F by polynomials.

Lemma 4.18 (Arakelyan's theorem)**.** *A closed set $F \subseteq \mathbb{C}$ is a Weierstrass set if and only if the following two conditions are satisfied:*

(K_1) $\hat{\mathbb{C}} \setminus F$ *is connected;*

(K_2) $\hat{\mathbb{C}} \setminus F$ *is locally connected at infinity.*

If in addition both the set F and the function $f \in A(f)$ are symmetric with respect to the real line, then the approximating function g can be chosen to be symmetric as well (see [Gau13, Section 2]).

Sometimes we may want to approximate a function in $A(f)$ so that the error is bounded by a given strictly positive function $\varepsilon : \mathbb{C} \to \mathbb{R}_+$ that is not constant, and $\varepsilon(z)$ may tend to zero as $z \to \infty$.

Definition 4.19 (Carleman set)**.** We say that a closed set $F \subseteq \mathbb{C}$ is a *Carleman set* in $\mathbb{C}$ if every function $f \in A(F)$ admits *tangential approximation* on F by entire functions; that is, for every strictly positive function $\varepsilon \in \mathcal{C}(F)$, there is an entire function g for which

$$|f(z) - g(z)| < \varepsilon(z) \quad \text{for all } z \in F.$$

It is clear that Carleman sets are a special case of Weierstrass sets and hence conditions (K_1) and (K_2) are necessary. Nersesyan's theorem gives sufficient conditions for tangential approximation [Ner71].

Lemma 4.20 (Nersesyan's theorem)**.** *A closed set F is a Carleman set in $\mathbb{C}$ if and only if conditions (K_1), (K_2) and*

(A) *for every compact set* $K \subseteq \mathbb{C}$ *there exists a neighbourhood* V *of infinity in* $\hat{\mathbb{C}}$ *such that no component of int* F *intersects both* K *and* V,

are satisfied.

Note that there is also a symmetric version of this result: if the set F and the functions f and ε are in addition symmetric with respect to $\mathbb{R}$ then the entire function g can be chosen to be symmetric with respect to $\mathbb{R}$ [Gau13, Section 2].

In some cases, depending on the geometry of the set F and the decay of the error function ε, we can perform tangential approximation on Weierstrass sets without needing condition (A); the next result can be found in [Gai87, Corollary in p.162].

Lemma 4.21. *Suppose* $F \subseteq \mathbb{C}$ *is a closed set satisfying conditions* (K_1) *and* (K_2) *that lies in a sector*

$$W_\alpha := \{z \in \mathbb{C} \;:\; |\arg z| \leqslant \alpha/2\},$$

for some $0 < \alpha \leqslant 2\pi$. *Suppose* $\tilde{\varepsilon}(t)$ *is a real function that is continuous and positive for* $t \geqslant 0$ *and satisfies*

$$\int_1^{+\infty} t^{-(\pi/\alpha)-1} \log \tilde{\varepsilon}(t) dt > -\infty.$$

Then every function $f \in A(F)$ *admits* ε*-approximation on the set* F *with* $\varepsilon(z) = \tilde{\varepsilon}(|z|)$ *for* $z \in F$.

4.5 CONSTRUCTION OF FUNCTIONS WITH WANDERING DOMAINS

To prove Theorem 4.1 we modify Baker's construction of a holomorphic self-map of $\mathbb{C}^*$ with a wandering domain escaping to infinity [Bak87, Theorem 4] to create instead a transcendental self-map of $\mathbb{C}^*$ with a wandering domain that accumulates to zero and to infinity according to a prescribed essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$ and with index $n \in \mathbb{Z}$.

Proof of Theorem 4.1. We construct two entire functions g and h using Nersesyan's theorem so that the function $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$, which is a transcendental self-map of $\mathbb{C}^*$, has the following properties:

- there is a bi-infinite sequence of annuli sectors $\{A_m\}_{m\in\mathbb{Z}\setminus\{0\}}$ that accumulate at zero and infinity and integers $s(m) \in \mathbb{Z}\setminus\{0\}$, for $m \in \mathbb{Z}\setminus\{0\}$, such that $f(A_m) \subseteq A_{s(m)}$ for all $m \in \mathbb{Z}$;
- the discs $B_+ := \overline{D(2, 1/4)}$ and $B_- := 1/B_+ = \overline{D(32/63, 4/63)}$ both map strictly inside themselves under f, $f(B_+) \subseteq \operatorname{int} B_+$ and $f(B_-) \subseteq \operatorname{int} B_-$;
- there is a bi-infinite sequence of closed discs $\{B_m\}_{m\in\mathbb{Z}\setminus\{0\}}$ such that $f(B_m) \subseteq \operatorname{int} B_+$, if $m > 0$, and $f(B_m) \subseteq \operatorname{int} B_-$, if $m < 0$.

Here $s(m) := \pi(\pi^{-1}(m) + 1)$ and the map $\pi : \mathbb{N} \longrightarrow \mathbb{Z}\setminus\{0\}$ is an ordering of the sets $\{A_m\}_{m\in\mathbb{N}}$ according to the sequence e; that is, $\pi(k)$ is the position of the kth component in the orbit of the wandering domain. More formally, we define

$$\pi(k) := \begin{cases} \#\{\ell \in \mathbb{N}_0 \;:\; e_\ell = \infty \text{ for } \ell < k\} + 1, & \text{if } e_k = \infty, \\ -\#\{\ell \in \mathbb{N}_0 \;:\; e_\ell = 0 \text{ for } \ell < k\} - 1, & \text{if } e_k = 0, \end{cases} \tag{4.4}$$

for $k \in \mathbb{N}$ (see Figure 16).

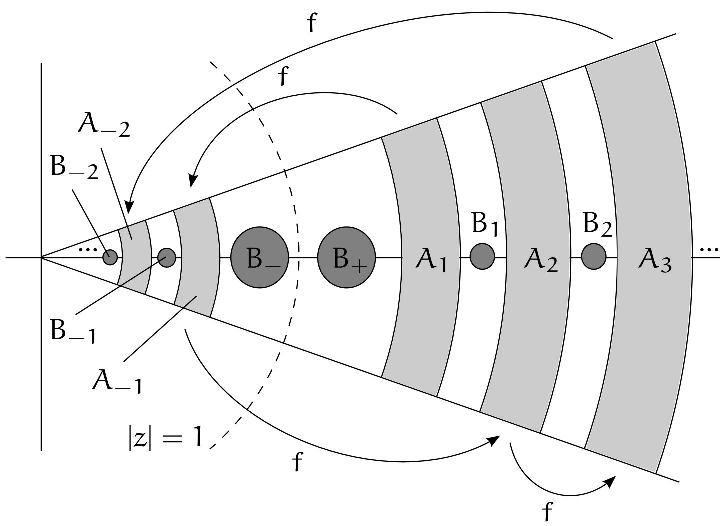


Figure 16: Sketch of the construction in the proof of Theorem 4.1.

By Montel's theorem, the domains $\{A_m\}_{m\in\mathbb{Z}\setminus\{0\}}$, $\{B_m\}_{m\in\mathbb{Z}\setminus\{0\}}$ and B_+, B_- are all contained in the Fatou set. Since $f(B_+) \subseteq \operatorname{int} B_+$, the function f has an attracting fixed point in B_+ and the sets $\{B_m\}_{m\in\mathbb{N}}$ are contained in the preimages of the immediate basin of attraction of this fixed point. Likewise, the sets $\{B_{-m}\}_{m\in\mathbb{N}}$ belong to the basin of attraction of an attracting fixed point in B_-. Observe that in order to show that A_1 is contained in a wandering domain that escapes following the essential itinerary e we need to prove that every A_m is contained in a different Fatou component.

Now let us construct the entire functions g and h so that the function $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ has the properties stated above. Note that in this construction $\log z$ denotes the principal branch of the logarithm with $-\pi < \arg z < \pi$. Let $0 < R < \pi/2$ and set, for $m > 0$, define

$$A_m := \{z \in \mathbb{C} \ : \ -R \leqslant \arg(z) \leqslant R,\ k_m \leqslant |z| \leqslant k_m e^{2R}\},$$
$$B_m := \overline{D\big((k_{m+1} - k_m)/2,\ 1/8\big)},$$

where k_m is any sequence of positive real numbers such that $k_m > 5/2$ and $k_{m+1} > k_m + 1/4$ for all $m \in \mathbb{N}$. We define $A_{-m} := 1/A_m$ and $B_{-m} := 1/B_m$ for all $m \in \mathbb{N}$. Note that $\log A_m$ is a square of side $2R$ centred at a point that we denote by $a_m \in \mathbb{R}$. Hence, $\log A_m$ contains the disc $D(a_m, R)$ for all $m \in \mathbb{Z}\setminus\{0\}$. The set

$$F := \overline{D(0,1)} \cup B_+ \cup \bigcup_{m>0} (A_m \cup B_m)$$

which consists of a countable union of disjoint compact sets is a Carleman set.

Let $\delta_+, \delta_- > 0$ be such that $|w - \ln 2| < \delta_+$ and $|w - \ln 32/63| < \delta_-$ imply, respectively, that $|e^w - 2| < 1/8$ and $|e^w - 32/63| < 2/63$. Let $K := \min\{R/4, \delta_\pm/4\}$. By Lemma 4.20, there is an entire function g that satisfies the following conditions:

$$\begin{cases} |g(z) - a_{s(m)} - n\log z| < R/4, & \text{if } z \in A_m \text{ with } m > 0, \\ |g(z) - \ln 2 - n\log z| < \delta_+/4, & \text{if } z \in \bigcup\limits_{m>0} B_m \cup B_+, \\ |g(z)| < K, & \text{if } z \in D(0,1), \end{cases}$$

Similarly, there is an entire function h that satisfies the following conditions:

$$\begin{cases} |h(z) - a_{s(-m)} - n\log(1/z)| < R/4, & \text{if } z \in A_m \text{ with } m > 0, \\ |h(z) - \ln 32/63 - n\log(1/z)| < \delta_-/4, & \text{if } z \in \bigcup\limits_{m>0} B_m \cup B_+, \\ |h(z)| < K, & \text{if } z \in D(0,1). \end{cases}$$

Therefore, since the sets B_- and A_m, $m < 0$, are contained in $D(0,1)$ and the sets B_+ and A_m, $m > 0$, are contained in $\mathbb{C} \setminus \overline{D(0,1)}$, the function $\log f(z) = g(z) + h(1/z) + n\log z$ satisfies

$$\begin{cases} |\log f(z) - a_{s(m)}| < R/2, & \text{if } z \in A_m \text{ with } m \neq 0, \\ |\log f(z) - \ln 2| < \delta_+/2, & \text{if } z \in \bigcup\limits_{m>0} B_m \cup B_+, \\ |\log f(z) - \ln 32/63| < \delta_-/2, & \text{if } z \in \bigcup\limits_{m<0} B_m \cup B_-, \end{cases}$$

and hence f has the required mapping properties.

Finally, note that this construction is symmetric with respect to the real line and hence all Fatou components of f that intersect the real line will be symmetric too. Thus, since transcendental self-maps of $\mathbb{C}^*$ cannot have doubly connected Fatou components that do not surround the origin [Bak87, Theorem 1], the Fatou components containing the sets $\{A_m\}_{m\in\mathbb{Z}\setminus\{0\}}$ are pairwise disjoint and $A_{\pi(0)}$ is contained in a wandering domain in $I_e(f)$. ■

4.6 CONSTRUCTION OF FUNCTIONS WITH BAKER DOMAINS

In this section we construct holomorphic self-maps of $\mathbb{C}^*$ with Baker domains. The construction is split into two cases: first, we deal with the cases that the function f is a transcendental entire or meromorphic function, that is, $f(z) = z^n \exp(g(z))$ where $n \in \mathbb{Z}$ and g is a non-constant entire function (see Theorem 4.24), and then we deal with the case that the function f is a transcendental self-map of $\mathbb{C}^*$, that is, $f(z) = z^n \exp(g(z) + h(1/z))$ where $n \in \mathbb{Z}$ and g, h are non-constant entire functions (see Theorem 4.2). For transcendental self-maps of $\mathbb{C}^*$, we are able to construct functions with Baker domains that have any given *periodic* essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$.

To that end, we use Lemma 4.21 to obtain entire functions g and, if necessary, h so that the function f has a Baker domain. After this approximation process, the resulting function f will behave as the function $T_\lambda(z) = \lambda z$, $\lambda > 1$, in a certain half-plane W. We first require the following result that estimates the asymptotic distance between the boundaries of $\log W$ and $\log T_\lambda(W) \subseteq \log W$.

Lemma 4.22. *Let $W = \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 2\}$ and, for $\lambda > 1$, let $T_\lambda(z) = \lambda z$. For $r > 0$, let $\delta(r)$ denote the vertical distance between the curves $\partial \log W$ and $\partial \log T_\lambda(W) \subseteq \log W$ along the vertical line $V_r := \{z \in \mathbb{C} : \operatorname{Re} z = r\}$. Then $\delta(r) \sim 2(\lambda - 1)e^{-r}$ as $r \to +\infty$.*

Proof. Since $\log z = \ln|z| + i\arg(z)$, the quantity $\delta(r)$ equals the difference between the arguments of the points z_1, z_2 with $\operatorname{Im} z_k > 0$, $k \in \{1, 2\}$, where the vertical lines ∂W and $\partial T(W)$ intersect the circle $\exp V_r$ of radius e^r (see Figure 17).

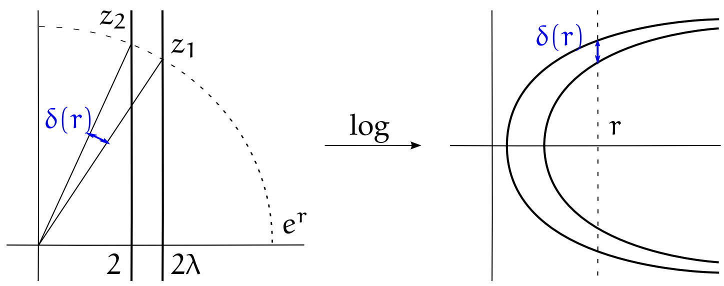


Figure 17: Definition of the function $\delta(r)$.

Since $\arg z_1, \arg z_2 \to \pi/2$ as $r \to +\infty$, we have

$$\delta(r) = \arccos\frac{2}{e^r} - \arccos\frac{2\lambda}{e^r} \sim \left(\frac{\pi}{2} - \frac{2}{e^r}\right) - \left(\frac{\pi}{2} - \frac{2\lambda}{e^r}\right) = \frac{2(\lambda - 1)}{e^r},$$

as $r \to +\infty$, as required. ∎

Given $N \in \mathbb{N}$ and a periodic sequence $e = \overline{e_0 e_1 \cdots e_{N-1}} \in \{0, \infty\}^{\mathbb{N}_0}$, let $p, q \in \mathbb{N}$ denote

$$\begin{aligned} p = p(e) &:= \#\{k \in \mathbb{N}_0 \ : \ e_k = \infty \text{ for } k < N\}, \\ q = q(e) &:= \#\{k \in \mathbb{N}_0 \ : \ e_k = 0 \text{ for } k < N\}, \end{aligned} \tag{4.5}$$

so that $p + q = N$. We want to construct a holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ with an N-cycle of Baker domains that has components U_i^∞, $0 \leqslant i < p$, and U_i^0, $0 \leqslant i < q$, in which

$$f^{Nn}_{|U_i^\infty} \to \infty \quad \text{and} \quad f^{Nn}_{|U_i^0} \to 0 \quad \text{locally uniformly as } n \to \infty.$$

In the case that zero is *not* an essential singularity of f, then $q = 0$ and $N = p$. Note that the closure of a Baker domain in $\hat{\mathbb{C}}$ may contain both zero and infinity.

For $p \in \mathbb{N}$ and $X \subseteq \mathbb{C}^*$, we define

$$\sqrt[p]{X} := \{z \in \mathbb{C}^* \ : \ z^p \in X, \ |\arg z| < \pi/p\}.$$

In order to construct a function with an N-periodic Baker domain that has p components around zero or infinity, we will semiconjugate the function T_λ that we want to approximate in the half-plane W by the pth root function:

$$\begin{tikzcd} W \arrow[r, "T_\lambda"] & W \\ \sqrt[p]{W} \arrow[u, "z^p"] \arrow[r, "T_{\lambda,p}"'] & \sqrt[p]{W}. \arrow[u, "z^p"'] \end{tikzcd}$$

Next we look at the effect of this semiconjugation on the function δ.

Lemma 4.23. *Let W and T_λ, $\lambda > 1$, be as in Lemma 4.22. For $p \in \mathbb{N}$ and $\lambda > 1$, define the function $T_{\lambda,p}(z) := \sqrt[p]{T_\lambda(z^p)}$ on $\sqrt[p]{W}$ and, for $r > 0$, let $\delta_p(r)$ denote the vertical distance between the curves $\partial \log \sqrt[p]{W}$ and $\partial \log T_{\lambda,p}(\sqrt[p]{W}) \subseteq \log \sqrt[p]{W}$ along the vertical line $V_r := \{z \in \mathbb{C} : \operatorname{Re} z = r\}$. Then $\delta_p(r) \sim 2(\lambda - 1)e^{-pr}/p$ as $r \to +\infty$.*

Proof. The function $z \mapsto z^p$ maps the circle of radius e^r to the circle of radius e^{pr} while the function $z \mapsto \sqrt[p]{z}$ divides the argument of points on that circle by p, so

$$\delta_p(r) = \frac{\delta(pr)}{p}$$

and hence, by Lemma 4.22, $\delta_p(r) \sim 2(\lambda - 1)e^{-pr}/p$ as $r \to +\infty$. ■

In the following theorem we construct transcendental entire or meromorphic functions that are self-maps of $\mathbb{C}^*$ and have Baker domains in which points escape to infinity. These functions are of the form $f(z) = z^n \exp(g(z))$ where $n \in \mathbb{Z}$ and g is a non-constant entire function.

Theorem 4.24. *For every $N \in \mathbb{N}$ and $n \in \mathbb{Z}$, there exists a holomorphic self-map f of $\mathbb{C}^*$ with $\operatorname{ind}(f) = n$ that is a transcendental entire function, if $n \geqslant 0$, or a transcendental meromorphic function, if $n < 0$, and has a cycle of hyperbolic Baker domains of period N.*

Proof. Let $\omega_N := e^{2\pi i/N}$ and define

$$V_m := \omega_N^m \sqrt[N]{W} \subseteq \mathbb{C} \setminus \overline{\mathbb{D}} \quad \text{for } 0 \leqslant m < N,$$

where W is the closed half-plane from Lemma 4.22. We denote by V the union of all V_m for $0 \leqslant m < N$, and let $R := \mathbb{R}_-$, if N is odd, or $R := \{z \in \mathbb{C}^* : \arg z = \pi(1 - 1/N)\}$, if N is even. Then put

$$d := \min\{(\sqrt[N]{2} - 1)/3,\ \operatorname{dist}(V, R)/4\}, \tag{4.6}$$

and define the closed connected set

$$B := \{z \in \mathbb{C} : \operatorname{dist}(z, V) \geqslant d \text{ and } \operatorname{dist}(z, R) \geqslant d\}, \tag{4.7}$$

which satisfies $B' := \overline{D(1, d)} \subseteq \operatorname{int} B$ (see Figure 18).

Observe that the closed set $F := B \cup V$ satisfies the hypothesis of Lemma 4.21; namely $\hat{\mathbb{C}} \setminus F$ is connected and $\hat{\mathbb{C}} \setminus F$ is locally connected at infinity, and $F \subseteq W_\alpha$ with $\alpha = 2\pi$. We now define a function $\hat{g}$ on F:

$$\hat{g}(z) := \begin{cases} \log\left(\omega_N^{m+1} \sqrt[N]{\lambda(z/\omega_N^m)^N}\right) - n \log z, & \text{for } z \in V_m,\ 0 \leqslant m < N, \\ -n \log z, & \text{for } z \in B, \end{cases} \tag{4.8}$$

where we have taken an analytic branch of the logarithm defined on $\mathbb{C}^* \setminus R$ and hence on F. Then $\hat{g} \in A(F)$.

For $r > 0$, we define the positive continuous function

$$\varepsilon(r) := \min\{d',\ k^{-(N+1)},\ r^{-(N+1)}\} \tag{4.9}$$

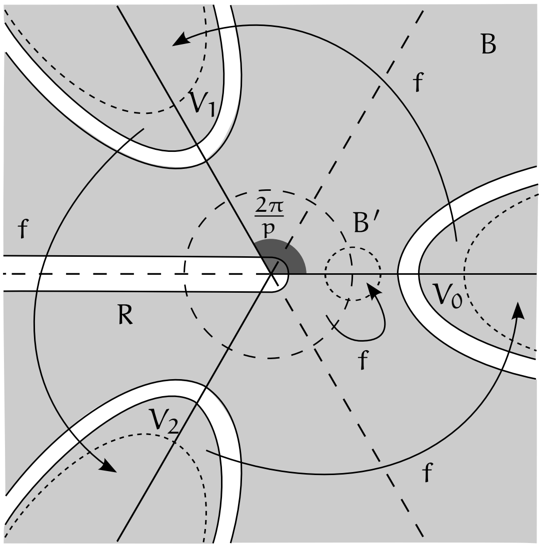


Figure 18: Sketch of the construction in the proof of Theorem 4.24 with $N=3$. The sets B and V_m, $0 \leqslant m < N$, are shaded in grey.

where the constant $d' > 0$ is so small that $|e^z - 1| < d$ for $|z| < d'$ and the constant $k > 0$ is so large that, for all $z \in \log T_\lambda(W)$ with $\operatorname{Re} z < k$, the disc $D(z, k^{-(N+1)})$ is compactly contained in $\log W$ and, moreover, if $\delta_N(r)$ is the function from Lemma 4.23, then

$$\varepsilon(r) < \delta_N(\ln(\lambda r)) \qquad \text{for } r \geqslant k, \tag{4.10}$$

which is possible since

$$\delta_N(\ln(\lambda r)) \sim \frac{2(\lambda - 1)}{N\lambda^N r^N} \qquad \text{as } r \to +\infty.$$

Since ε satisfies

$$\int_1^{+\infty} r^{-3/2} \ln \varepsilon(r) dt = C - (N+1) \int_{r_0'}^{+\infty} \frac{\ln r}{r^{3/2}} dr > -\infty$$

for some constants $C \in \mathbb{R}$ and $r_0' \geqslant r_0$, by Lemma 4.21 (with $\alpha = 2\pi$), there is an entire function g such that

$$|g(z) - \hat{g}(z)| < \varepsilon(|z|) \qquad \text{for all } z \in F. \tag{4.11}$$

We put

$$f(z) := z^n \exp(g(z)) = z^n \exp(\hat{g}(z)) \exp(g(z) - \hat{g}(z)). \tag{4.12}$$

By Lemma 4.23 and (4.8-4.15), $f(V_m) \subseteq V_{m+1}$ for $0 \leqslant m < N-1$ and $f(V_{N-1}) \subseteq V_0$ and, by (4.6-4.15), $f(B) \subseteq D(1, d)$. Hence each set V_m is contained in an N-periodic Fatou component U_m for $0 \leqslant m < N$ and B is contained in the immediate basin of attraction of an attracting fixed point that lies in B'. It follows that the Fatou components U_m are all simply connected.

To conclude the proof of Theorem 4.24, it only remains to check that the Fatou components U_m, $0 \leqslant m < N$, are hyperbolic Baker domains. Due to symmetry, it suffices to deal with the case $m = 0$. Let $z_0 \in U_0$. Since $V_0 \subseteq U_0$ is an absorbing region, we can assume without loss of generality that $z_0 \in V_0$ and $|z_0|$ is sufficiently large. For $n \in \mathbb{N}$, let

$$\epsilon_n := g(f^{n-1}(z_0)) - \hat{g}(f^{n-1}(z_0))$$

which, by (4.15), satisfies

$$|\epsilon_n| < \varepsilon(|f^{n-1}(z_0)|) \quad \text{as } n \to \infty.$$

For $n \in \mathbb{N}$, define

$$C_n := \prod_{0<k\leqslant n} \exp \epsilon_k = \exp \sum_{0<k\leqslant n} \epsilon_k,$$

which represents the quotient $f^n(z_0)/\big(z^n \exp(\hat{g}(z_0))\big)$. Using the triangle inequality, we obtain

$$|C_n| \leqslant \exp \sum_{0<k\leqslant n} |\epsilon_k| < \exp \sum_{0<k\leqslant n} \varepsilon(|f^{k-1}(z_0)|). \tag{4.13}$$

Next, we are going to show that $|C_n|$ is bounded above for all $n \in \mathbb{N}$. To that end, we find a lower bound for $|f^k(z_0)|$ for $k \in \mathbb{N}$ assuming, if necessary, that $|z_0| = r_0$ is sufficiently large. Put $K := (\sqrt[N]{\lambda} - 1)/2 > 0$. Then $|C_1| > 1/K$ for $r_0 > 0$ sufficiently large and, by (4.12) and (4.8),

$$|f(z_0)| = \sqrt[N]{\lambda}|z_0||C_1| \geqslant \frac{\sqrt[N]{\lambda}}{K} r_0 = \mu r_0,$$

with $\mu := \sqrt[N]{\lambda}/K > 1$. Hence, by induction and the symmetry properties of the sets V_m, $0 \leqslant m < N$,

$$|f^k(z_0)| \geqslant \mu^k r_0 \quad \text{for } k \in \mathbb{N}. \tag{4.14}$$

In particular, $z_0 \in I(f)$ so, by normality, the periodic Fatou components U_m, $0 \leqslant m < N$, are Baker domains. We deduce by (4.13), (4.9) and (4.14) that $|C_n| < e^S$ for all $n \in \mathbb{N}$, where $S < +\infty$ is the sum of the following geometric series

$$S := \sum_{k=0}^{\infty} \frac{1}{(\mu^k r_0)^{N+1}} = \frac{1}{r_0^{N+1}} \sum_{k=0}^{\infty} \left(\frac{1}{\mu^{N+1}}\right)^k = \frac{\mu^{N+1}}{r_0^{N+1}(\mu^{N+1}-1)}.$$

Next we use the characterisation of Lemma 4.11 to show that the Baker domains are hyperbolic. For $n \in \mathbb{N}$, define

$$c_n = c_n(z_0) = \frac{|f^{(n+1)N}(z_0) - f^{nN}(z_0)|}{\operatorname{dist}(f^{nN}(z_0), \partial U)}.$$

We have

$$f^{nN}(z_0) = C_{nN} \sqrt[N]{\lambda^{nN} z_0^N} = C_{nN} \lambda^n z_0 \quad \text{for } n \in \mathbb{N}$$

and therefore

$$|f^{(n+1)N}(z_0) - f^{nN}(z_0)| \sim C_\infty \lambda^n (\lambda - 1)|z_0| \quad \text{as } n \to \infty,$$

where $C_\infty := \lim_{n\to\infty} C_n$. Also, $\operatorname{dist}(f^{nN}(z_0), \partial U_0) \leqslant e^S \lambda^n |z_0|$ and hence if $c := (\lambda-1)/2 > 0$, we have $c_n(z_0) > c$ for all $n \in \mathbb{N}$. Thus, by Lemma 4.11, the Baker domain U_0 is hyperbolic. This completes the proof of Theorem 4.24. ■

Finally we prove Theorem 4.2 in which we construct a function f that is a transcendental self-map of $\mathbb{C}^*$ with $\operatorname{ind}(f) = n$ that has a cycle of hyperbolic Baker domains in $I_e(f)$, where e is any prescribed periodic essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$.

Proof of Theorem 4.2. Let $N \in \mathbb{N}$ be the period of e and let $p, q \in \mathbb{N}_0$ denote, respectively, the number of symbols 0 and ∞ in the sequence

$e_0e_1\dots e_{N-1}$, where $p+q = N$; see (4.5). We modify the proof of Theorem 4.24 to obtain a transcendental self-map of $\mathbb{C}^*$ of the form

$$f(z) := z^n \exp(g(z)z^{N+1} + h(1/z)/z^{N+1})$$

that has a hyperbolic Baker domain U in $I_e(f)$, where the entire functions g, h will be constructed using approximation theory.

We start by defining a collection of p sets $\{V_m^\infty\}_{0\leqslant m<p}$, whose closure in $\hat{\mathbb{C}}$ contains infinity. Put $\omega_p := e^{2\pi i/p}$ once again and define

$$V_m^\infty := \omega_p^m \sqrt[p]{W} \subseteq \mathbb{C}\setminus\overline{D(0,\rho)} \quad \text{for } 0\leqslant m<p,$$

where W is the half-plane from Lemma 4.22 and $\rho := 1 + (\sqrt[N]{2}-1)/6$. We denote by V_∞ the union of all V_m^∞, $0\leqslant m<p$.

As before, we define a set B_∞ that will be contained in an immediate basin of attraction of f and put $R_\infty = \mathbb{R}_-$, if p is odd, or $R_\infty = \{z\in\mathbb{C}^* : \arg z = \pi(1-1/p)\}$, if p is even. Then, let

$$d_\infty := \min\{(\sqrt[N]{2}-1)/6,\ \operatorname{dist}(V_\infty, R_\infty)/4\},$$

and define the closed connected set

$$B_\infty := \{z\in\mathbb{C} : \operatorname{dist}(z, V_\infty) \geqslant d_\infty \text{ and } \operatorname{dist}(z, R_\infty) \geqslant d_\infty\}\setminus D(0,\rho),$$

which compactly contains the disc $B'_\infty := \overline{D((1+\sqrt[N]{2})/2, (\sqrt[N]{2}-1)/6)}$. Finally, we define the disc $D := D(0, 1/\rho)$, which is contained in $\mathbb{D}$. We will construct the function g by approximating it on the closed set $F_\infty := V_\infty \cup B_\infty \cup D$, which satisfies the hypothesis of Lemma 4.21; namely $\hat{\mathbb{C}}\setminus F_\infty$ is connected and $\hat{\mathbb{C}}\setminus F_\infty$ is locally connected at infinity, and $F_\infty \subseteq W_\alpha$ with $\alpha = 2\pi$ (see Figure 19).

Similarly, we define a set B_0 and a collection of q unbounded sets $\{V_m^0\}_{0\leqslant m<q}$ by using the same procedure as above, just replacing p by q, and then, if V_0 is the union of all V_m^0, $0\leqslant m<q$, we put $F_0 := V_0\cup B_0\cup D$. The Fatou set of the function f will contain all the sets V_m^∞, $0\leqslant m<p$, and all the sets $\tilde{V}_m^0 := 1/V_m^0$, $0\leqslant m<q$, which are unbounded in $\mathbb{C}^*$.

In order to define the functions $\hat{g}\in A(F_\infty)$ and $\hat{h}\in A(F_0)$, we first introduce some notation to describe how $\hat{g}$ and $\hat{h}$ map the components of V_∞ and V_0, respectively; we use the same notation as in

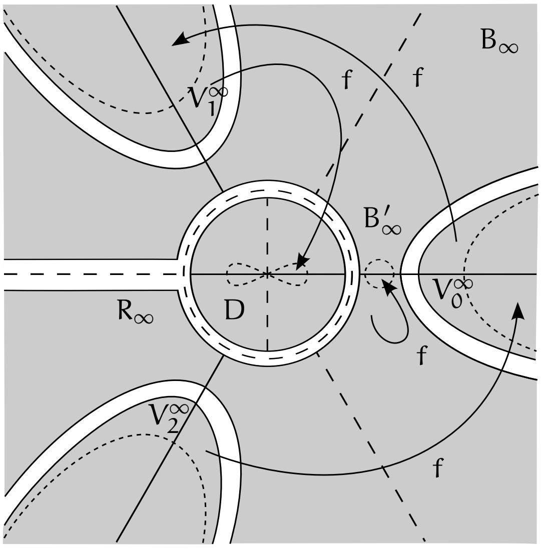


Figure 19: Sketch of the construction of the entire function g in the proof of Theorem 4.2 with $e = \overline{\infty\infty 0 0 \infty}$. The sets D, B_∞ and V_m^∞, $0 \leqslant m < p$, are shaded in grey.

Theorem 4.1. Let $\pi : \{0, \dots, N-1\} \to \{-q, \dots, -1, 1, \dots, p\}$ denote the function given by, for $0 \leqslant k < N$,

$$\pi(k) := \begin{cases} \#\{\ell \in \mathbb{N}_0 \ : \ e_\ell = \infty \text{ for } \ell < k\} + 1, & \text{if } e_k = \infty, \\ -\#\{\ell \in \mathbb{N}_0 \ : \ e_\ell = 0 \text{ for } \ell < k\} - 1, & \text{if } e_k = 0. \end{cases}$$

The function π is an ordering of the components of $V_\infty \cup 1/V_0$ according to the sequence e. Suppose that V is the starting component; that is, $V = \tilde{V}_0^0$, if $e_0 = 0$, and $V = V_0^\infty$, if $e_0 = \infty$. Then

$$f^k(V) \subseteq \begin{cases} V^\infty_{\pi(k)}, & \text{if } \pi(k) > 0, \\ \tilde{V}^0_{-\pi(k)}, & \text{if } \pi(k) < 0. \end{cases}$$

For $m \in \{-q, \dots, -1, 1, \dots, p\}$, we define the function

$$s(m) := \pi(\pi^{-1}(m) + 1 \pmod N),$$

which describes the image of the component V_m^∞, if $m > 0$, and $\tilde{V}_m^0$, if $m < 0$, so that the function f to be constructed has a Baker domain that has essential itinerary e. More formally, for $0 \leqslant m < p$,

$$f(V_m^\infty) \subseteq \begin{cases} V^\infty_{s(m)}, & \text{if } s(m) > 0, \\ \tilde{V}^0_{-s(m)}, & \text{if } s(m) < 0; \end{cases}$$

and, for $0 \leqslant m < q$,

$$f(\tilde{V}_m^0) \subseteq \begin{cases} V_{s(-m)}^\infty, & \text{if } s(-m) > 0, \\ \tilde{V}_{-s(-m)}^0, & \text{if } s(-m) < 0. \end{cases}$$

We now give the details of the construction of the entire function g from the function $\hat{g} \in A(F_\infty)$. For $z \in V_m^\infty$, $0 \leqslant m < p$, we put

$$\hat{g}(z) := \begin{cases} \left(\log\left(\omega_p^{s(m)} \sqrt[p]{\lambda(z/\omega_p^m)^p}\right) - n\log z\right)/z^{N+1}, & \text{if } s(m) > 0, \\ \left(\log\left(\omega_p^{s(m)} / \sqrt[p]{\lambda(z/\omega_p^m)^p}\right) - n\log z\right)/z^{N+1}, & \text{if } s(m) < 0, \end{cases}$$

for $z \in B_\infty$, we put $\hat{g}(z) := (\log(1 + (\sqrt[n]{2} - 1)/2) - n\log z)/z^{N+1}$ and, for $z \in D$, we put $\hat{g}(z) := 0$, where we have taken an analytic branch of the logarithm defined on $\mathbb{C}^* \setminus R_\infty$ and hence on $V_\infty \cup B_\infty$ (see Figure 19). Then $\hat{g} \in A(F_\infty)$. For $r > 0$, we define the positive continuous function ε_∞ by

$$\varepsilon_\infty(r) := \min\{d'_\infty,\ k_\infty^{-(N+1)},\ r^{-(N+1)}\}/(2r^{N+1})$$

where the constant $d'_\infty > 0$ is so small that $|e^z - 1| < d_\infty$ for $|z| < d'_\infty$ and the constant $k_\infty > 0$ is so large that, for all $z \in \log T_\lambda(W)$ with $\operatorname{Re} z < k_\infty$, the disc $D(z, k_\infty^{-(N+1)})$ is compactly contained in $\log W$ and, moreover, if $\delta_N(r)$ is the function from Lemma 4.23, then

$$\varepsilon_\infty(r) \cdot 2r^{N+1} < \delta_N(\ln(\lambda r)) \quad \text{for } r \geqslant k_\infty,$$

which, as before, is possible since

$$\delta_N(\ln(\lambda r)) \sim \frac{2(\lambda - 1)}{N\lambda^N r^N} \quad \text{as } r \to +\infty.$$

Since ε_∞ satisfies

$$\int_1^{+\infty} r^{-3/2} \ln \varepsilon_\infty(r)\,dt > -\infty,$$

by Lemma 4.21 (with $\alpha = 2\pi$), there is an entire function g such that

$$|g(z) - \hat{g}(z)| < \begin{cases} \varepsilon_\infty(|z|) & \text{for } z \in V_\infty \cup B_\infty, \\ 1/2 & \text{for } z \in D. \end{cases} \tag{4.15}$$

Similarly, we can construct an entire function h that approximates a function $\hat{h} \in A(F_0)$ so that the function

$$\begin{aligned} f(z) &:= z^n \exp(g(z)z^{N+1} + h(1/z)/z^{N+1}) \\ &= z^n \exp(\hat{g}(z)z^{N+1}) \exp(\hat{h}(1/z)/z^{N+1}) \cdot \\ &\quad \cdot \exp((g(z) - \hat{g}(z))z^{N+1}) \exp((h(z) - \hat{h}(z))/z^{N+1}) \end{aligned}$$

has the desired properties. Observe that if $z \in V_\infty \cup B_\infty$, then $1/z \in D$ and if $1/z \in V_0 \cup B_0$, then $z \in D$. Thus, $\hat{h}(1/z) = 0$ for $z \in V_\infty \cup B_\infty$ and

$$\begin{aligned} &|\hat{h}(1/z)/z^{N+1} + (g(z) - \hat{g}(z))z^{N+1} + (h(z) - \hat{h}(z))/z^{N+1}| \leqslant \\ &\qquad \leqslant 0 + 1/(2|z|^{N+1}) + 1/(2|z|^{N+1}) = 1/|z|^{N+1} \end{aligned}$$

for $z \in V_\infty \cup B_\infty$. Therefore, by Lemma 4.23, each component of the set $V_\infty \cup 1/V_0$ is contained in an N-periodic Fatou component and the sets B_∞ and $1/B_0$ are contained in the immediate basins of attraction of two attracting fixed points that lie in B'_∞ and B'_0, respectively (see Figure 20; here $\tilde{B}_0 = 1/B_0$ and $\tilde{B}'_0 = 1/B'_0$).

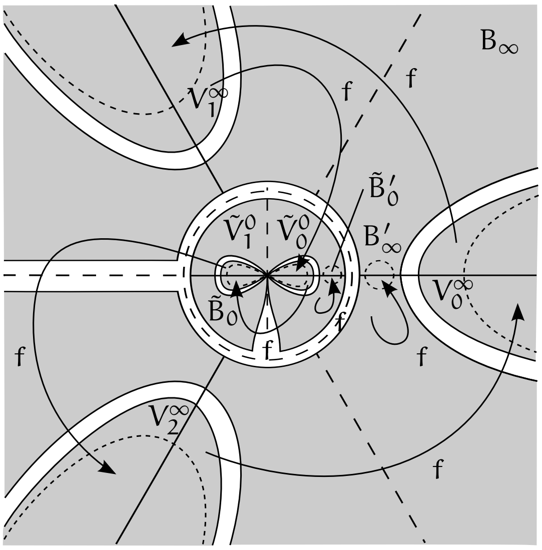


Figure 20: Sketch of the construction of the function f in the proof of Theorem 4.2 with $e = \overline{\infty\infty 00\infty}$. The sets B_∞, $\tilde{B}_0$ and V_m^∞, $0 \leqslant m < p$, and $\tilde{V}_m^0$, $0 \leqslant m < q$, are shaded in grey.

Finally, a similar argument to that in the proof of Theorem 4.24 shows that the Fatou components that we have constructed are hyperbolic Baker domains; we omit the details. ∎

5 QUESTIONS FOR FURTHER RESEARCH

Rådström [Råd53] solved the question of what was the most general class of holomorphic self-maps to which the main results of the iteration theory of Fatou and Julia could be extended. However, the theory of Fatou and Julia can be extended to the iteration of functions that are not holomorphic self-maps. For instance, the iteration of transcendental meromorphic functions has been widely studied (see [Ber93]). In this chapter we describe the research done on the escaping set of meromorphic functions with several essential singularities which would be a natural continuation of this thesis.

5.1 ITERATION OF MEROMORPHIC FUNCTIONS

We say that f is a *transcendental meromorphic function* (in $\mathbb{C}$) if f has an essential singularity at infinity and is holomorphic in $\mathbb{C}$ except for a discrete set of *poles* $B_0(f) := f^{-1}(\infty) \subseteq \mathbb{C}$; see [Ber93] for a survey on the iteration of meromorphic functions. Perhaps the best known example of a transcendental meromorphic function is the tangent function, which has poles at the odd multiples of $\pi/2$. The main difference between the iteration of transcendental entire functions and that of transcendental meromorphic functions is the existence of points in the set

$$B(f) := \bigcup_{n \in \mathbb{N}} f^{-n}(\infty),$$

which have a truncated orbit under iteration by f. Note that $B(f)$ is always countable.

Let f be a transcendental meromorphic function that is not entire. If infinity is an exceptional value of f, then since meromorphic functions have at most two exceptional values, f has a single pole that is omitted and hence is conjugated to a holomorphic self-map of $\mathbb{C}^*$ of the form $f(z) = z^n \exp(g(z))$ where $n < 0$ and g is a non-constant entire function; we discussed the iteration of such functions in Chapter 1.

Otherwise, f has at least two poles or one pole that is not omitted, and we denote this set of functions by $\mathcal{M}_\infty$. However, observe that, unlike for entire functions, the set $\mathcal{M}_\infty$ is not closed under composition; we shall discuss this fact in the next section.

Baker, Kotus and Lü studied the iteration of meromorphic functions in the series of papers [BKL91a; BKL90; BKL91b; BKL92]. For functions $f \in \mathcal{M}_\infty$, we define the Fatou set $F(f)$ in the usual way but adding the requirement that for $z \in F(f)$, $f^n(z)$ is defined for all $n \in \mathbb{N}$. Then, since $B(f)$ contains infinitely many points (in fact, the set $f^{-2}(B_0)$ is already infinite), it follows from Montel's theorem that $J(f) = \mathbb{C} \setminus F(f) = \overline{B(f)}$. Recall that $\mathcal{O}^-(z_0, f) = \{z \in \mathbb{C} \ : \ z_0 = f^n(z)$ for some $n \in \mathbb{N}\}$. In [BKL91a], the authors showed that if $z_0 \in \mathbb{C}$ is not an exceptional value, then $J(f) \subseteq \mathcal{O}^-(z_0, f)'$ and hence $J(f)$ is perfect and $J(f) = B(f)'$.

The iteration of $T_\lambda(z) := \lambda \tan z$, $\lambda \in \mathbb{R} \setminus \{0\}$, was considered for the first time by Devaney and Keen [DK89] who showed that either $J(T_\lambda) = \mathbb{R}$, if $|\lambda| \geqslant 1$, or $J(T_\lambda) \subseteq \mathbb{R}$ is a Cantor set, if $0 < |\lambda| < 1$. Later, Keen and Kotus [KK97] studied the dynamics of T_λ for $\lambda \in \mathbb{C}^*$ (see also [Jia91]).

We can also consider the iteration of transcendental meromorphic functions in $\mathbb{C}^*$, that is, functions that have two essential singularities, at zero and infinity, and are holomorphic in $\mathbb{C}^*$ except for discrete sets of zeros and poles. Such functions are to transcendental self-maps of $\mathbb{C}^*$ what transcendental meromorphic functions are to transcendental entire functions. Examples of these functions are given by

$$f(z) = m(z + 1/z) \quad \text{ or } \quad g(z) = m(z)\tilde{m}(1/z)$$

where m and $\tilde{m}$ are transcendental meromorphic functions (in $\mathbb{C}$). If f is a transcendental meromorphic function in $\mathbb{C}^*$ that has a finite number of zeros and poles, then f is of the form

$$f(z) = R(z) \exp(g(z) + h(1/z))$$

where R is a rational function and g, h are non-constant entire functions. The iteration of transcendental meromorphic functions in $\mathbb{C}^*$ should be related to the iteration of transcendental self-maps of $\mathbb{C}^*$ that we study in this thesis, in the same way that the iteration of transcendental meromorphic functions (in $\mathbb{C}$) is related to the iteration of transcendental entire functions. We plan to study the escaping set of such functions in the course of studying more general classes of meromorphic functions, described in the next section.

5.2 THE WORKS OF BOLSCH AND HERRING

In general, the composition of two transcendental meromorphic functions f, g is not a transcendental meromorphic function because the poles of the first function f become essential singularities of the composition $g \circ f$. Note that these points may not be isolated singularities, so here we are using the term *essential singularity* in a wider sense to denote any obstruction to analytic continuation that is not a pole.

Far less is known about the iteration of meromorphic functions with more than one essential singularity. In this direction, in their theses, Bolsch [Bol97] (supervised by Prof. Pommerenke) and Herring [Her95] (supervised by Prof. Baker) proposed, independently, two generalisations of transcendental meromorphic functions with several essential singularities for which the Fatou and Julia theory extends, with appropriate modifications.

Bolsch's class $\mathcal{K}$

Bolsch introduced the following class of meromorphic functions

$$\mathcal{K} := \left\{ f \ : \ \begin{array}{c} \text{there is a compact countable set } E(f) \subseteq \hat{\mathbb{C}} \text{ such that} \\ f \text{ is meromorphic in } \hat{\mathbb{C}} \setminus E(f) \text{ but in no proper superset} \end{array} \right\},$$

which is the smallest class that is closed under composition and contains the set of transcendental meromorphic functions (in $\mathbb{C}^*$) that we denoted by $\mathcal{M}_\infty$ earlier. Note that originally Bolsch denoted this class by $\mathcal{S}$ but this could be confused with the *Speiser class* of finite type transcendental entire functions, so instead we follow the notation from [BDH01].

In [Bol96], the author showed that if $f \in \mathcal{K}$ is not a Möbius transformation, then the repelling cycles of f are dense in $J(f)$. Later on, in [Bol99], Bolsch studied how Fatou components map to each other under iteration by meromorphic functions and, for $f \in \mathcal{K}$, proved that if U and U' are Fatou components of f such that $f(U) \subseteq U'$, then $U' \setminus U$ contains at most two points (see also [Her98]). He also proved that, more generally, if $E(f)$ is a set of capacity zero, then the Fatou components of f are simply, doubly or infinitely connected.

Herring's class $\mathcal{M}$

Herring studied the iteration of what he called meromorphic functions *outside a small set*. He introduced the class

$$\mathcal{M} := \left\{ f \ : \ \begin{array}{c} \text{there is a compact totally disconnected set } E(f) \subseteq \hat{\mathbb{C}} \\ \text{such that } f \text{ is meromorphic in } \hat{\mathbb{C}} \setminus E(f) \text{ and} \\ \text{the cluster set } C(f, \hat{\mathbb{C}} \setminus E(f), \alpha) = \hat{\mathbb{C}} \text{ for all } \alpha \in E(f); \\ \text{if } E(f) = \emptyset, \text{ then } f \text{ is neither constant nor univalent} \end{array} \right\}.$$

If $E \subseteq \hat{\mathbb{C}}$ is a compact totally disconnected set and $f : \hat{\mathbb{C}} \setminus E \to \hat{\mathbb{C}}$ is a meromorphic function, we define the *cluster set* of f at $\alpha \in E$ with respect to the set $\hat{\mathbb{C}} \setminus E$ by

$$C(f, \hat{\mathbb{C}} \setminus E, \alpha) := \left\{ w \in \hat{\mathbb{C}} \ : \ \begin{array}{c} \text{there is } (z_n) \subseteq \hat{\mathbb{C}} \setminus E \text{ such that} \\ z_n \to \alpha \text{ and } f(z_n) \to w \text{ as } n \to \infty \end{array} \right\}.$$

The class $\mathcal{M}$ is closed under composition and contains the class $\mathcal{K}$. For every compact totally disconnected set $E \subseteq \hat{\mathbb{C}}$ it is possible to construct a function $f \in \mathcal{M}$ with $E(f) = E$ and, under some additional hypothesis on $E \subseteq \hat{\mathbb{C}}$, every function that is meromorphic in $\hat{\mathbb{C}} \setminus E$ and has essential singularities at every point of E is in the class $\mathcal{M}$ (see [BDH01; BDH04]).

Herring also studied the following subclasses of $\mathcal{M}$ consisting of functions that satisfy a k-*Picard property* and that are closed under composition. For $k \geqslant 2$, he defined

$$\mathcal{MP}_k := \left\{ f \in \mathcal{M} \ : \ \begin{array}{c} E(f) \neq \emptyset \text{ and for every } \alpha \in E(f) \text{ and} \\ \text{open set } U \ni \alpha, \ \#\hat{\mathbb{C}} \setminus f(U \setminus E(f)) \leqslant k \end{array} \right\},$$

note that $\mathcal{K} \subset \mathcal{MP}_2$. Finally, some of the results from [BDH01] were proven for the following class of finite-type functions

$$\mathcal{MS} := \{f \in \mathcal{M} \ : \ \operatorname{sing}(f^{-1}) \text{ is finite}\}$$

that is the analogue of the Speiser class $\mathcal{S}$. They adapted the techniques from [EL92] to show that functions in $\mathcal{MS}$ have no Baker domains and they also proved a 'no wandering domains' theorem for a subclass of $\mathcal{MS}$.

5.3 THE ESCAPING SET OF A MEROMORPHIC FUNCTION

Domínguez [Dom98] defined the escaping set of a transcendental meromorphic function f by

$$I(f) := \{z \in \mathbb{C} \ : \ f^n(z) \neq \infty \text{ for all } n \in \mathbb{N} \text{ and } f^n(z) \to \infty \text{ as } n \to \infty\}$$

and proved that the analogues of Eremenko's properties (I1) and (I2) hold in this setting, namely

$$I(f) \cap J(f) \neq \emptyset \quad \text{ and } \quad J(f) = \partial I(f).$$

She also observed that property (I3) does not necessarily hold, that is, the components of $\overline{I(f)}$ may be bounded even in the case that f has a single pole.

For a transcendental meromorphic function f in $\mathbb{C}^*$, we can define the escaping set by

$$I(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \notin \{0,\infty\} \text{ for all } n \in \mathbb{N} \text{ and } \omega(z,f) \subseteq \{0,\infty\}\}$$

and, similarly, we can adapt Definition 2.2 to define sets $I_e(f) \subseteq I(f)$, for $e \in \{0,\infty\}^{\mathbb{N}_0}$. It would be interesting to study the properties of such sets and, in particular, see whether analogues of properties (I1) and (I2) hold in this setting as well.

Regarding the iteration of meromorphic functions with several essential singularities, very little is known about the escaping set. If

$f : \hat{\mathbb{C}} \setminus E(f) \to \hat{\mathbb{C}}$ is a meromorphic function and $E(f)$ consists of essential singularities of f, then we define

$$I(f) := \{z \in \hat{\mathbb{C}} \setminus E(f) \ : \ f^n(z) \notin E(f) \text{ for all } n \in \mathbb{N} \text{ and } \omega(z, f) \subseteq E(f)\}.$$

There is no mention of $I(f)$ in the works of Bolsch. In [BDH01], the authors considered the following subsets of $I(f)$ for functions f in the class $\mathcal{M}$. For $\alpha \in E(f)$, they defined

$$I(f, \alpha) := \{z \in \hat{\mathbb{C}} \setminus E(f) : f^n(z) \notin E(f) \text{ for } n \in \mathbb{N} \text{ and } f^n(z) \to \alpha \text{ as } n \to \infty\}$$

and, for $f \in \mathcal{MP}_k$, $k \geqslant 2$, proved that these sets satisfy the properties (I1) and (I2):

$$J(f) \cap I(f, \alpha) \neq \emptyset \quad \text{ and } \quad J(f) = \partial I(f, \alpha).$$

Note that if $E(f) = \{0, \infty\}$ and f is a transcendental self-map of $\mathbb{C}^*$, then $I(f, 0) = I_0(f)$ and $I(f, \infty) = I_\infty(f)$. However, the approach that we follow in this thesis takes into consideration every possible way of accumulating to $E(f)$. It would be interesting to investigate whether the notion of essential itinerary can be adapted to study the escaping set of functions in the classes $\mathcal{K}$ or $\mathcal{M}$ and see if Eremenko's properties (I1) and (I2) hold for more general subsets of $I(f)$.

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INDEX

NOTATION

(a_n)	Sequence given by $a_0, a_1, a_2, \ldots$
$A(f)$	Fast escaping set of a function f (see Section 1.2)
$A_e(f)$	Set of fast escaping points of a transcendental self-map of $\mathbb{C}^*$ with essential itinerary e (see Definition 2.3)
$A_e^{-\ell,k}(f,R)$	Level of the fast escaping set $A_e(f)$ (see Definition 2.3)
$A(F)$	Class of continuous functions $f : F \to \mathbb{C}$ that are holomorphic in $\operatorname{int} F$
$AV(f)$	Set of asymptotic values of a function f (see Section 1.1)
$AV_\alpha(f)$	Set of asymptotic values of a transcendental self-map f of $\mathbb{C}^*$ with asymptotic path to $\alpha \in \{0,\infty\}$ (see Section 3.2)
$\mathcal{B}$	Eremenko-Lyubich class of bounded-type transcendental entire functions (see Section 1.2)
$\mathcal{B}^*$	Class of bounded-type transcendental self-maps of the punctured plane (see Section 3.1)
$B(f)$	Set of poles and prepoles of a transcendental meromorphic function f (see Section 5.1)
$\mathbb{C}$	Complex plane
$\hat{\mathbb{C}}$	Riemann sphere $\mathbb{C} \cup \{\infty\}$
$\mathbb{C}^*$	Punctured plane $\mathbb{C} \setminus \{0\}$
$C(f,X,\alpha)$	Cluster set of a function f at a point α with respect to a set X (see Section 5.2)
$\mathcal{C}(X)$	Class of continuous functions $f : X \to \mathbb{C}$
$\mathbb{C}[z]$	Set of polynomials in z with complex coefficients

$\mathrm{CP}(f)$	Set of critical points of a function f (see Section 1.1)
$\mathrm{CV}(f)$	Set of critical values of a function f (see Section 1.1)
$\mathbb{D}$	Unit disc $\{z \in \mathbb{C} \,:\, \lvert z\rvert < 1\}$
$\operatorname{dist}(X, Y)$	Euclidean distance between two sets $X, Y \subseteq \mathbb{C}$
$D(z_0, R)$	Disc of radius $R > 0$ centered at z_0, $\{z \in \mathbb{C} \,:\, \lvert z - z_0\rvert < R\}$
$E(f)$	Set of essential singularities of a function $f \in \mathcal{M}$
$E_\lambda(z)$	Exponential family $E_\lambda(z) = \lambda \exp z$ for $\lambda \in \mathbb{C}^*$
$\tilde{f}$	Lift of a function f ($\tilde{f}$ is not unique)
f^{-1}	Inverse of a function f
$f \sim g$	$f(r) \sim g(r)$ as $r \to +\infty$ means that $f(r)/g(r) \to 1$ as $r \to +\infty$
f^n	nth iterate of a function f given by $f \circ \overset{n}{\cdots} \circ f$ for $n \in \mathbb{N}$
$f_{\lvert X}$	Restriction of a function f to a set X
$F(f)$	Fatou set of a function f (see Section 1.1)
$\mathbb{H}$	Right half-plane $\{z \in \mathbb{C} \,:\, \operatorname{Re} z > 0\}$
$\mathbb{H}_R$	Right half-plane $\{z \in \mathbb{C} \,:\, \operatorname{Re} z > R\}$ for $R > 0$
$\mathbb{H}_R^{\pm}$	Set $\{z \in \mathbb{C} \,:\, \lvert\operatorname{Re} z\rvert > R\}$ for $R > 0$ consisting of the union of a left half-plane and a right half-plane
$I(f)$	Escaping set of a function f (see Section 1.2)
$I_e(f)$	Set of escaping points of a transcendental self-map of $\mathbb{C}^*$ with essential itinerary e (see Definition 2.2)
$I_e^{-\ell,k}(f)$	Subset of the escaping set $I_e(f)$ consisting of points z such that $f^\ell(z)$ has essential itinerary $\sigma^k(e)$ (see Definition 2.2)
$\operatorname{int} X$	Interior of a set X
$\operatorname{Im} z$	Imaginary part of a point $z \in \mathbb{C}$
$J(f)$	Julia set of a function f (see Section 1.1)
$\mathcal{K}$	Bolsch's class of meromorphic functions outside a closed countable set of essential singularities (see Section 5.2)

$M(r, f)$	Maxmimum modulus of a function f on the circle of radius $r > 0$, also written $M(r)$
$m(r, f)$	Minimum modulus of a function f on the circle of radius $r > 0$, also written $m(r)$
$\mathcal{M}$	Herring's class of meromorphic functions outside a small set of essential singularities (see Section 5.2)
$\mathcal{MP}_k$	Class of functions in $\mathcal{M}$ that satisfy a k-Picard property (see Section 5.2)
$\mathcal{MS}$	Class of finite-type functions in $\mathcal{M}$ (see Section 5.2)
$\mathcal{M}_\infty$	Class of transcendental meromorphic functions (in $\mathbb{C}$) (see Section 5.1)
$\mathbb{N}$	Set of natural numbers $1, 2, 3, \ldots$
$\mathbb{N}_0$	Set of non-negative integers $\mathbb{N} \cup \{0\}$
$O(g(r))$	$f(r) = O(g(r))$ as $r \to +\infty$ means that there is $C > 0$ such that $\|f(r)\| \leqslant C\|g(r)\|$ for all $r > 0$ sufficiently large
$\mathcal{O}^+(z_0, f)$	Forward orbit of a point z_0 under iteration by a function f
$\mathcal{O}^-(z_0, f)$	Backward orbit of a point z_0 under iteration by a function f
$\mathcal{O}(z_0, f)$	Grand orbit of a point z_0 under iteration by a function f
$\mathbb{Q}$	Set of rational numbers p/q with $p, q \in \mathbb{Z}$
$P(f)$	Postsingular set of a function f (see Section 3.2)
$\operatorname{Re} z$	Real part of a point $z \in \mathbb{C}$
$\mathcal{S}$	Speiser class of finite-type transcendental entire functions (see Section 1.1)
$S(f)$	Set of singular values of a function f (see Section 1.1)
$\operatorname{sing}(f^{-1})$	Set of inverse function singularities of a function f (see Section 1.1)
$T_\lambda(z)$	Tangent family $T_\lambda(z) = \lambda \tan z$ for $\lambda \in \mathbb{C}^*$
$\overline{X}$	Closure of a set X in $\mathbb{C}$

$\hat{X}$	Closure of a set X in $\hat{\mathbb{C}}$
$X^{\mathbb{N}_0}$	Set of all sequences of elements of a set X
∂X	Boundary of a set X
$\partial_{\text{in}} X$	Inner boundary of a doubly connected set X
$\partial_{\text{out}} X$	Outer boundary of a doubly connected set X
$\mathbb{Z}$	Set of integer numbers $\ldots, -1, 0, 1, \ldots$
$\lambda(f)$	Lower order of an entire function f (see Section 3.1)
$\rho(f)$	Order of an entire function f (see Section 3.1)
$\rho_\alpha(f)$	Order of a transcendental self-map f of $\mathbb{C}^*$ at $\alpha \in \{0, \infty\}$ (see Definition 3.30)
$\rho_\Omega(z)$	Hyperbolic density in a domain Ω
$\omega(z_0, f)$	Omega-limit set of the orbit of a point z_0 under iteration by a function f
$\#X$	Cardinality of a set X
$[z, w]_\Omega$	Hyperbolic distance from z to w in a domain Ω