From polygons to polyhedra and beyond

ST PAUL’S GEOMETRY MASTERCLASS I

Who are we?

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- Final year maths PhD student at The Open University
- Studying links between geometry and numbers
- Also interested in the history of maths

David Martí Pete

- Second year PhD student at The Open University
- Studying complex dynamics
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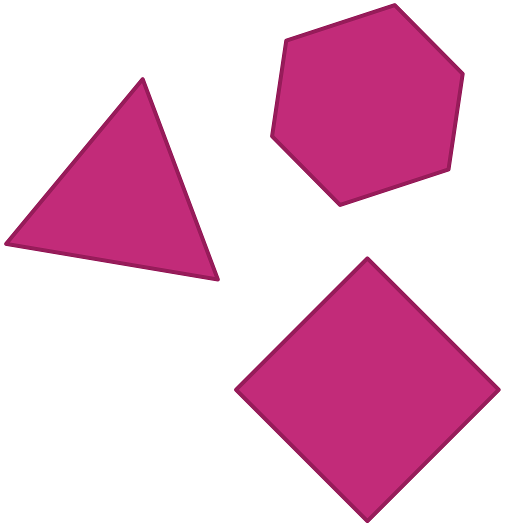
What are we doing?

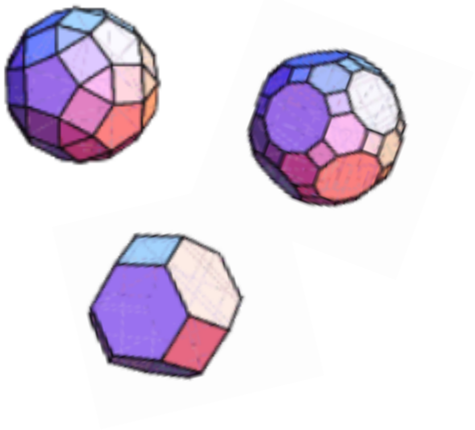
We have organised a series of workshops to show you what it’s like to study maths at university. We’ve based the themes on aspects of our own research, and some of our favourite topics!

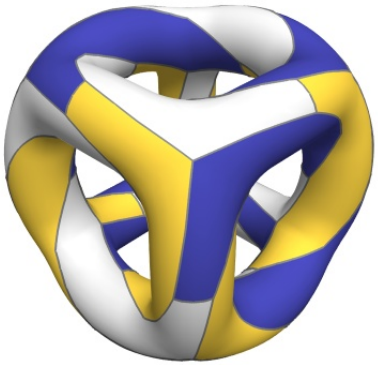
The workshops are:

- From polygons to polyhedra and beyond
- Fractals: Friends or foe?
- Mapping the world

From polygons to polyhedra and beyond

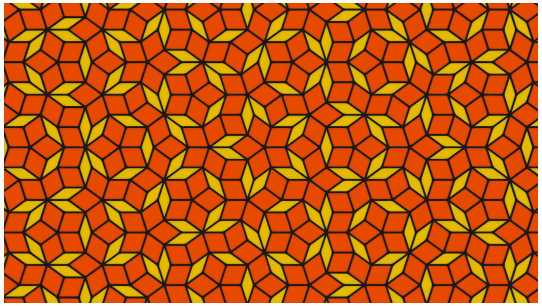


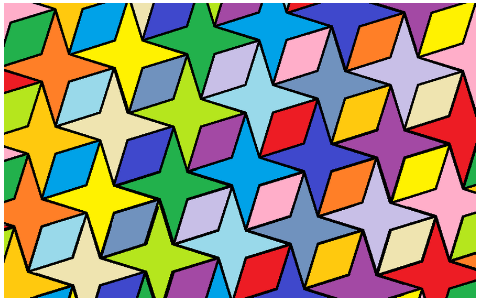


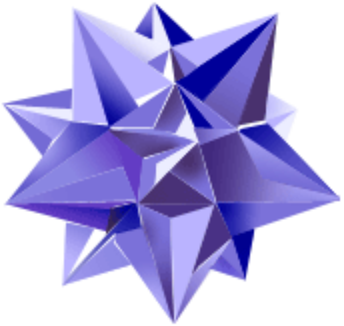


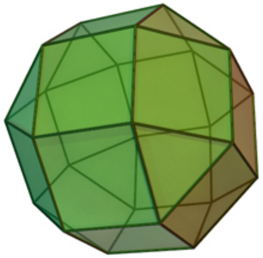
Joining polygons

Polygons can be joined together to make all different kinds of shapes.







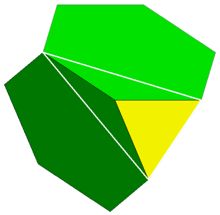


From polygons to polyhedra

To keep things simple we will construct polyhedra:

- Out of **regular** polygons
- So that the same arrangement of faces can be found at each **vertex**.

A recipe tells you which faces to join at each vertex i.e. (3,6,6) means place a triangle followed by two hexagons (in a clockwise order) at each vertex.



From polygons to polyhedra

Recipe	Polyhedra?	Tessellation?	Nothing?
(6,6,6)			
(3,6,6)			
(4,6,8)			
(3,6,8)			
(3,3,4,3,4)			
(4,4,4)			
(3,3,3,3,3)			
(4,5,5)			
(4,8,8)			

A mathematical problem

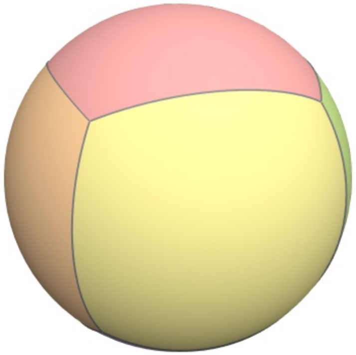
Polyhedra have been studied by mathematicians for thousands of years.

- How can we classify them?
- How many of each type are there?
- How do we know we have found them all?

We will try to answer these questions for some of the polyhedra we've found today.

Polyhedra: tessellations of a sphere

We can view polyhedra as tessellations of a sphere.



Euler's formula says that for any tessellation of a sphere, with V vertices, E edges and F faces,

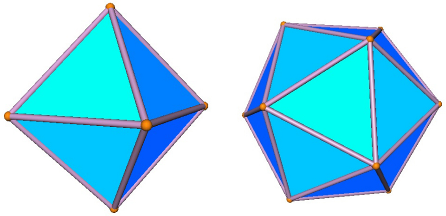
$$\boxed{V-E+F=2}$$

The Platonic solids

The Platonic solids are those polyhedra where:

- Every face has the same number of edges
- The same number of edges meet at each vertex.

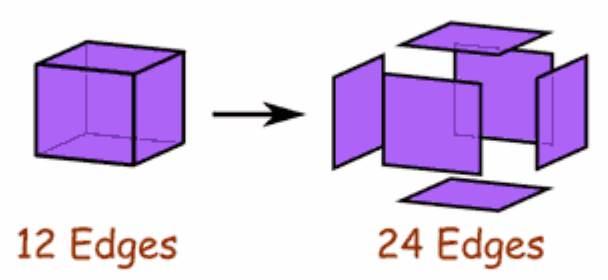
Examples include the octahedron and the icosahedron.



How many are there in total?

Proof I

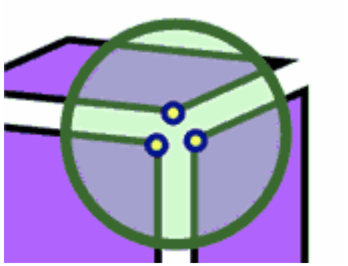
Say that every face has p edges/vertices, and q faces meet at each vertex. We expand our polygon by cutting it along each edge. This doubles the number of edges to get 2E edges.



The new number of edges is also the number of faces times the number of edges per face: pF. So pF=2E.

Proof II

The number of vertices in the expanded shape is equal to the number of edges in it, so 2E.



The number of vertices in the expanded shape is also equal to the number of vertices in the original shape times the number of faces meeting at each vertex: qV. So 2E=qV.

Proof III

V-E+F=2 and **pF=2E=qV**

$2E/q - E + 2E/p = 2$

So

$1/q - 1/2 + 1/p = 1/E$

So

$1/q + 1/p = 1/E + 1/2 > 1/2$

The Platonic solids

$1/q + 1/p > 1/2$

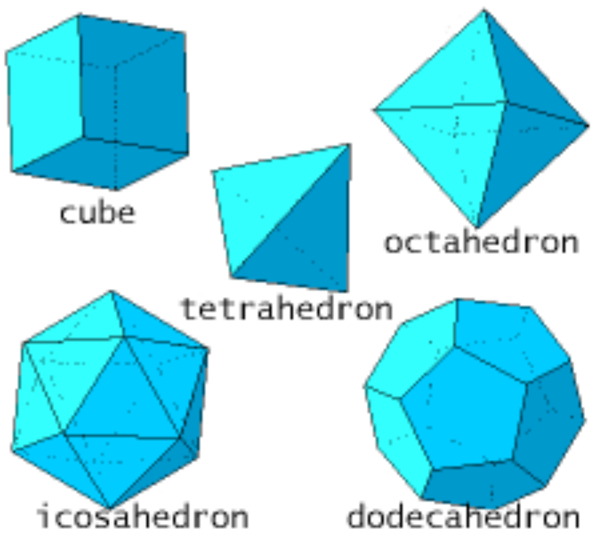
How many possibilities for p and q are there?

There are only 5!

{3,3}, {4,3}, {3,4}, {5,3}, {3,5}

(tetrahedron, octahedron, cube, icosahedron, dodecahedron)

The Platonic solids



We found all of the Platonic solids earlier!

Beyond polyhedra

When mathematicians solve one problem they often try to make it more general.

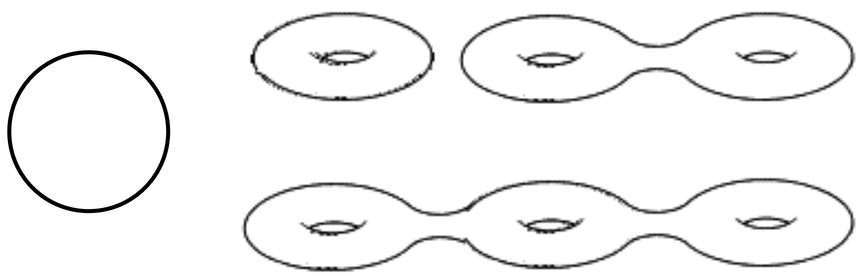
Mathematicians have studied polyhedra on spheres for thousands of years. But what about polyhedra cut out from other 3D shapes?

What other shapes might work? The flat plane won't do! We need to look for mathematical **surfaces**.

Surfaces

A surface is a shape that when you zoom in really close, it looks flat. A sphere is a surface. A jagged mountain top isn't as it has points!

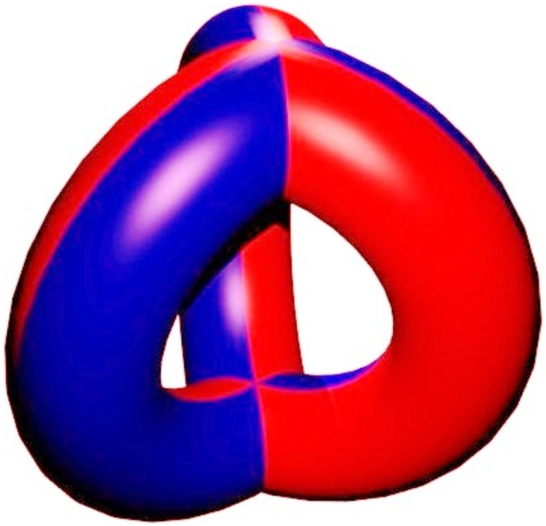
Actually, every single surface, once it's been stretched or squashed a bit, will look like one of these:



The number of holes (0,1,2,3,…) is called the genus of the surface.

Maps on surfaces

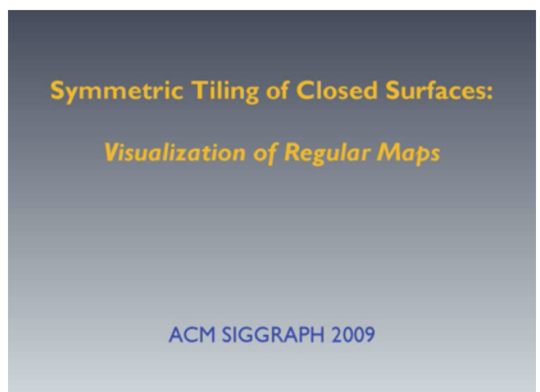
Just as we drew tessellations on the sphere to represent polyhedra, we can draw tessellations on other surfaces. We call these mathematical maps.



Generalising the Platonic solids

The Platonic solids are maps on a sphere for which every face has p edges/vertices and q faces meet. We want our generalisation to have this property too! We call them {p,q}-patterns:

A **{p,q}-pattern** is a map on a surface for which each face has p edges/vertices, and q faces meet at each vertex.



Finding the possibilities

We can adapt Euler's formula:

$$V-E+F=2-2g$$

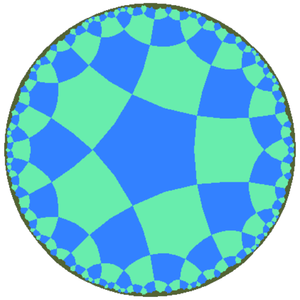
For the torus (genus 1) we can prove that there are only 3 possibilities.

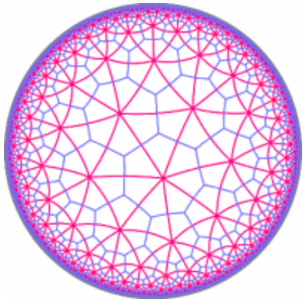
For higher genus there are more, but there can never infinitely many!

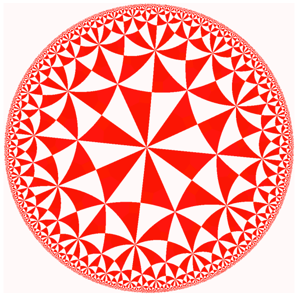
But how many actually exist?

But Euler's formula only says which values of p and q we can possibly have. It doesn't say that such a map will actually exist!

To solve this problem requires studying surfaces in a special way. The maps are transformed into tessellations of the hyperbolic plane





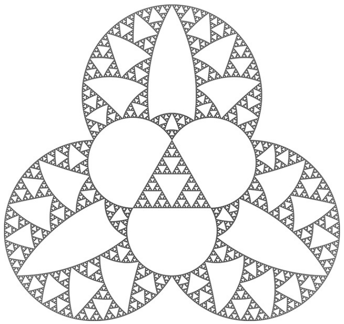


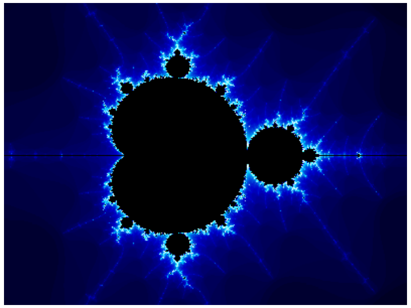
Then computers are needed to calculate from the millions of possibilities which ones will actually work!

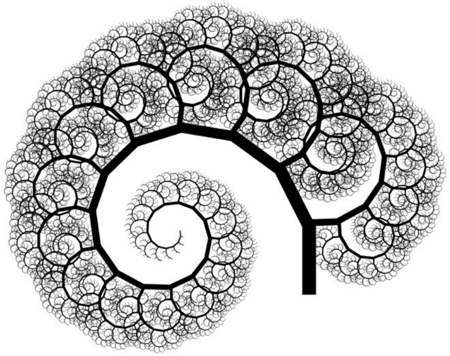
Next time…

How long is the coastline of Great Britain? Sounds a simple question, right, but it’s not actually so easy to answer!

Next week you’ll look at problems like this, and see how it relates to the theory of fractals, special shapes like these.







You’ll see how these shapes are neither 2D nor 3D, but something in between. Studying these fractal dimensions shows that the coastline of Great Britain is actually infinitely long…

Thanks to

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