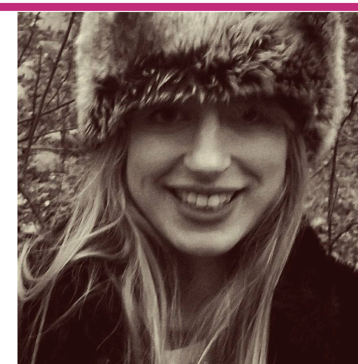
Fractals everywhere

ST PAUL'S GEOMETRY MASTERCLASS II

Who are we?

Mairi Walker

- Final year maths PhD student at The Open University
- Studying links between geometry and numbers
- Also interested in the history of maths



David Martí Pete

- Second year PhD student at The Open University
- Studying complex dynamics

What are we doing?

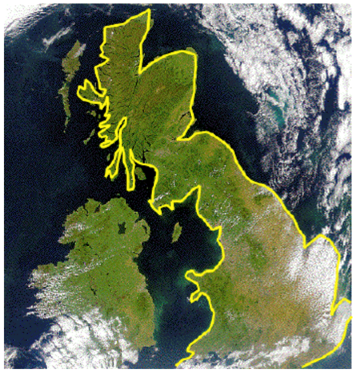
We have organised a series of workshops to show you what it's like to study maths at university. We've based the themes on aspects of our own research, and some of our favourite topics!

The workshops are:

- From polygons to polyhedra and beyond
- Fractals everywhere
- Mapping the world

How long is the coastline of Great Britain?

You can measure the length of a coastline using a map and string. Using the map's scale you can convert the length of the coastline on the map to the actual length of the coastline.



Map scale	Length of string (cm)	Length of coastline (km)
1:26,300,000	14.6	3829.8
1:11,100,000	36.6	4062.6
1:5,260,000	88.9	4676.1
1:4,170,000	125	5212.5

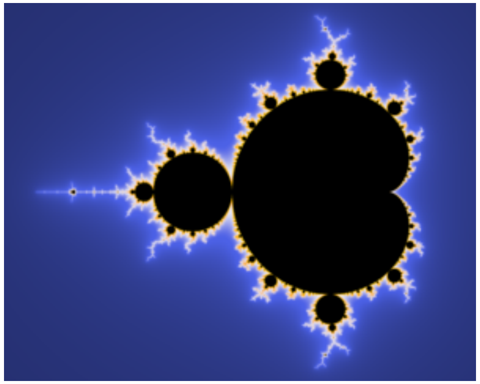
The coastline paradox

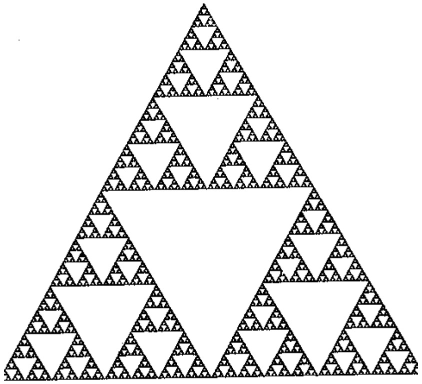
The **coastline paradox** is the observation that the length of a coastline is ambiguous, or, in mathematical terms, not well-defined; the length of a coastline depends on the scale at which you measure it, and increases without limit as the scale increases.

The coastline paradox happens because coastlines are examples of **fractals**, mathematical shapes or objects that exhibit a repeating pattern that displays at every scale.

Self-similarity

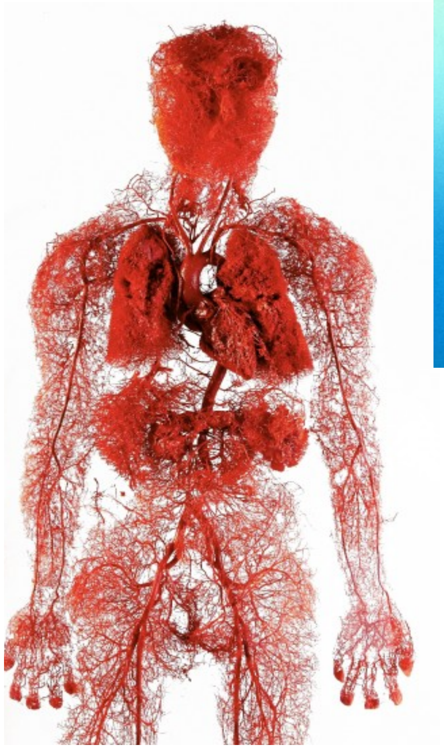
An object is **self-similar** if parts of it look sort of like the whole thing. If parts of it look exactly like the whole thing only smaller, then the object is **strictly self-similar**.





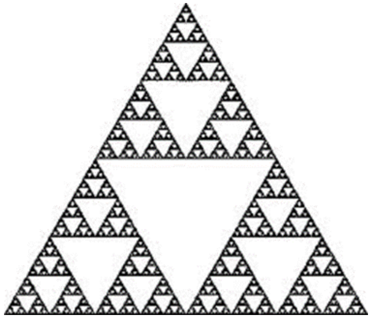
An object has **structure at all scales** if you can zoom in to any point, infinitely far, and keep finding more and more detail.

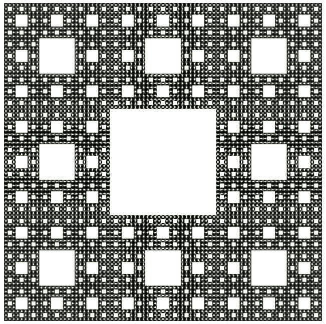
Fractals in nature

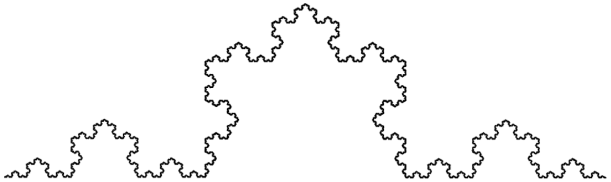


Geometric fractals

They are **strictly self-similar**!

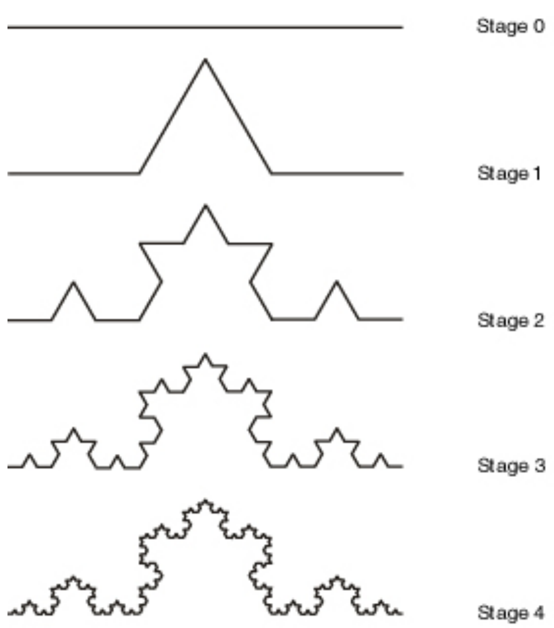






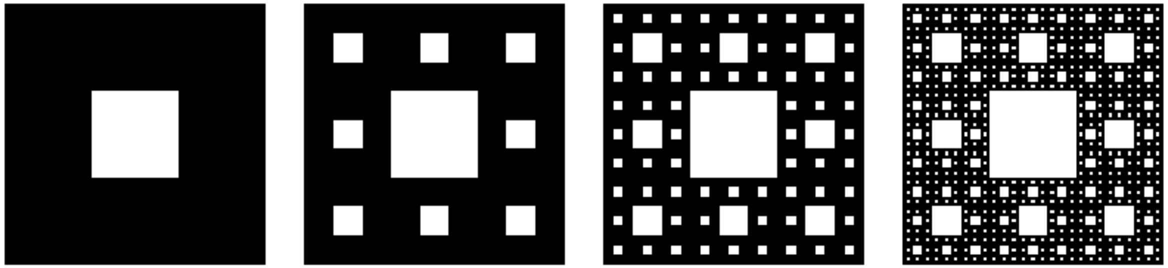
Constructing geometric fractals

They can be created using a recursive rule.



"Draw an equilateral triangle in the middle third of each straight line then remove the base."

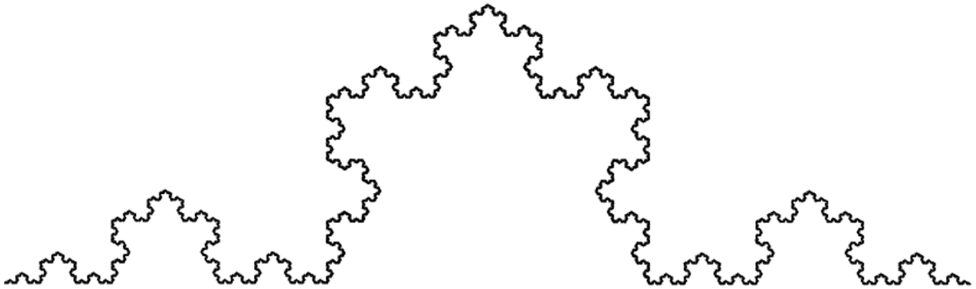
Constructing geometric fractals



"Divide each black square into 9 smaller squares, and remove the middle one"

Measuring self-similar fractals

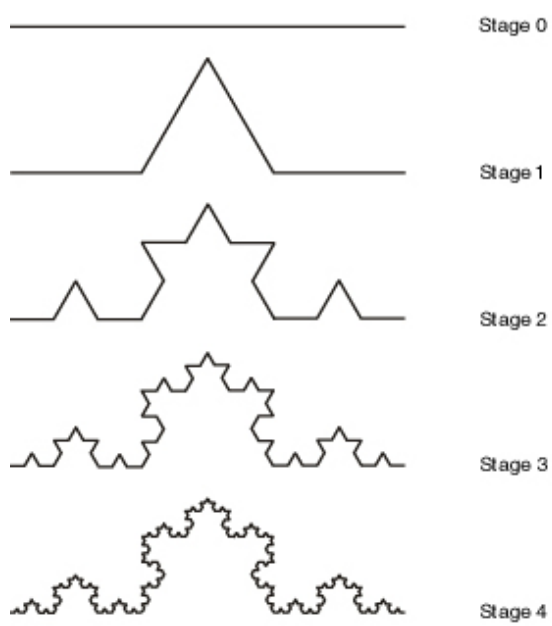
How long is this fractal, the von Koch curve?



Fractals have infinitely fine detail so are very difficult to measure!

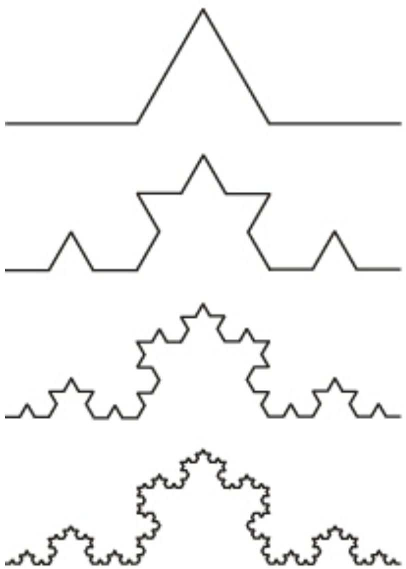
Measuring self-similar fractals

We can measure a self-similar fractal by measuring each stage of its construction.



The stages get closer and closer to the actual fractal, so their lengths should get closer and closer to the length of the actual fractal.

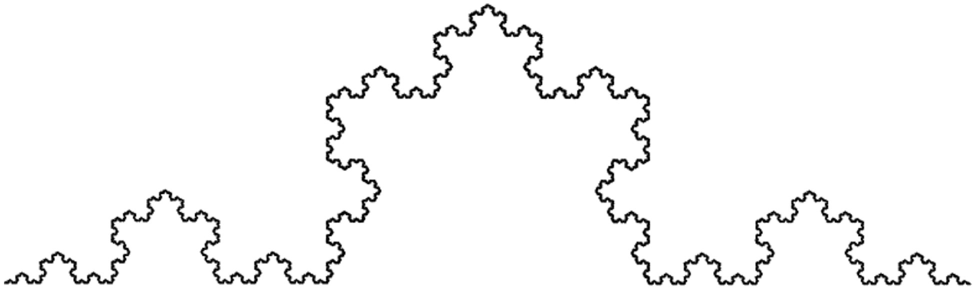
The length of the von Koch curve



Stage	Number of segments	Length of segments	Total length
1	4	$1/3$	$4/3$
2	$16=4\uparrow 2$	$1/9=(1/3)\uparrow 2$	$(4/3)\uparrow 2$
3	$64=4\uparrow 3$	$1/27=(1/3)\uparrow 3$	$(4/3)\uparrow 3$
n	$4\uparrow n$	$(1/3)\uparrow n$	$(4/3)\uparrow n$

The length of the von Koch curve

How long is this fractal, the von Koch curve?



It is infinitely long!

Measuring fractals

Funny things can happen when we measure fractals.

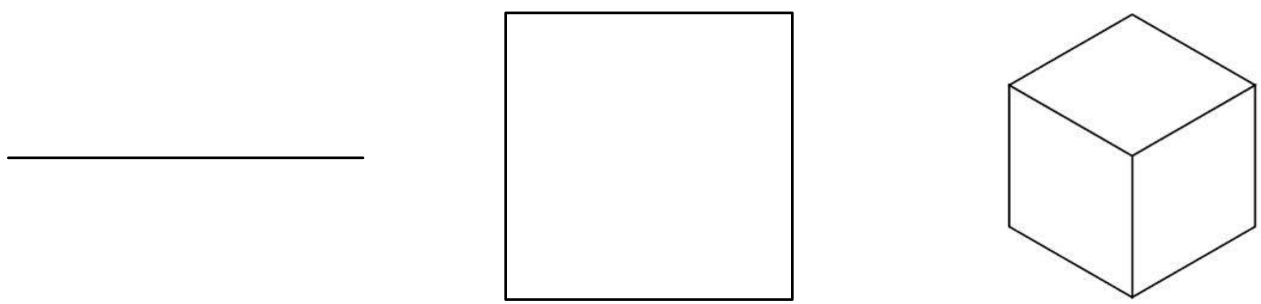
- The von Koch curve is infinitely long, yet does not stretch infinitely far in any one direction.
- The Sierpinski carpet has zero area, yet we can still see it.

Why do fractals have these strange properties?

It is to do with dimension!

Dimension

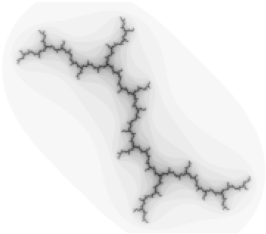
Lines have one dimension, planes have two dimensions, and our space has three dimensions.



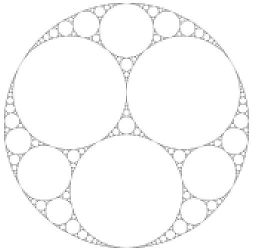
What dimension might a fractal have?

Fractal dimension

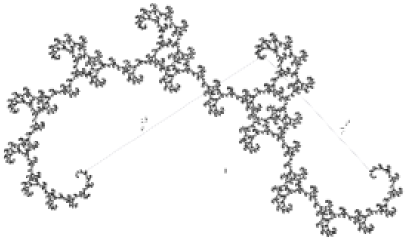
The dimension of a fractal is not generally a whole number!



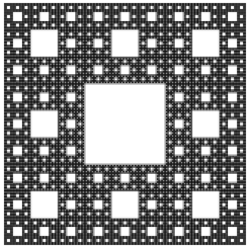
Dimension 1.2



Dimension 1.3057



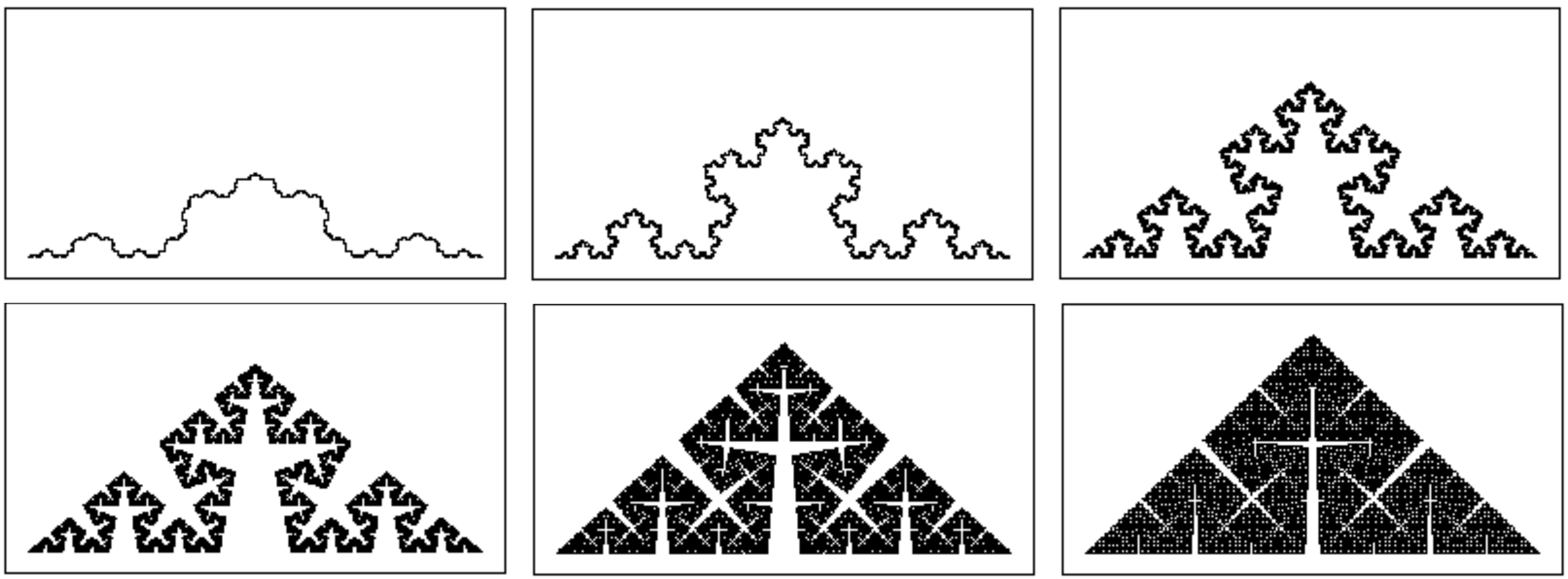
Dimension 1.61803



Dimension 1.8928

Fractal dimension

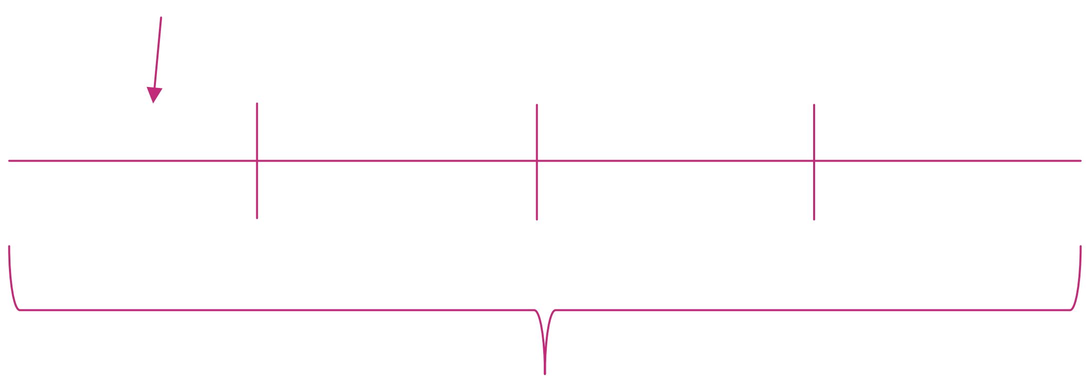
The closer the dimension is to two, the more solid the fractal appears.



Relating dimension to measurements

A line is made up of $1/r$ copies of itself scaled by a factor of r.

Scale factor $1/4$

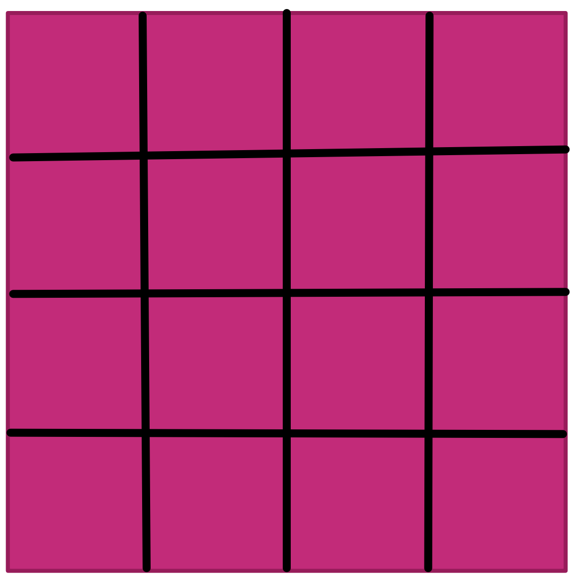


4 copies

Relating dimension to measurements

A square is made up of $(1/r)^2$ copies of itself scaled by a factor of r.

Scale factor $1/4$



16 copies

Relating dimension to measurements

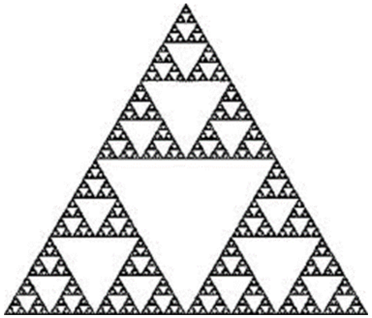
- A line is made up of $1/r$ copies of itself scaled by a factor of r.
- A square is made up of $(1/r)^2$ copies of itself scaled by a factor of r.
- A cube is made up of $(1/r)^3$ copies of itself scaled by a factor of r.

We have the formula

$N=(1/r)^d$.

Fractal dimension

$N = (1/r)^d$.



$r = 1/2$, $N = 3$

What is the dimension? We need to solve the equation for d.

A formula for fractal dimension

$N = (1/r)^d$.

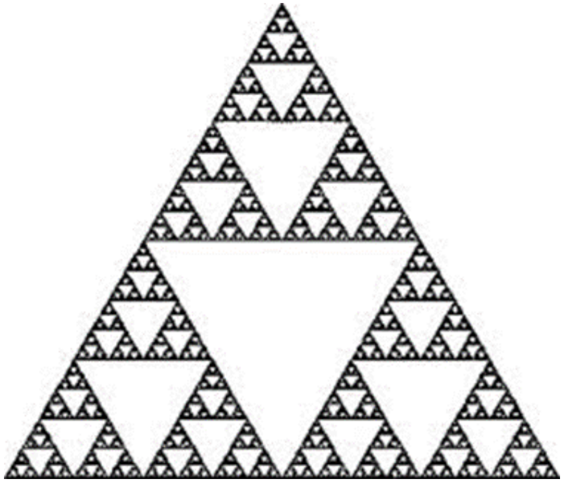
Take logs on both sides:

$\log(N) = \log((1/r)^d) = d\log(1/r)$.

Giving the equation

$d = \log(N) / \log(1/r)$.

The dimension of the Sierpinski triangle



$d = \log(N) / \log(1/r)$

$r = 1/2$, $N = 3$

$d = \log(3) / \log(2) = 1.58496\ldots$

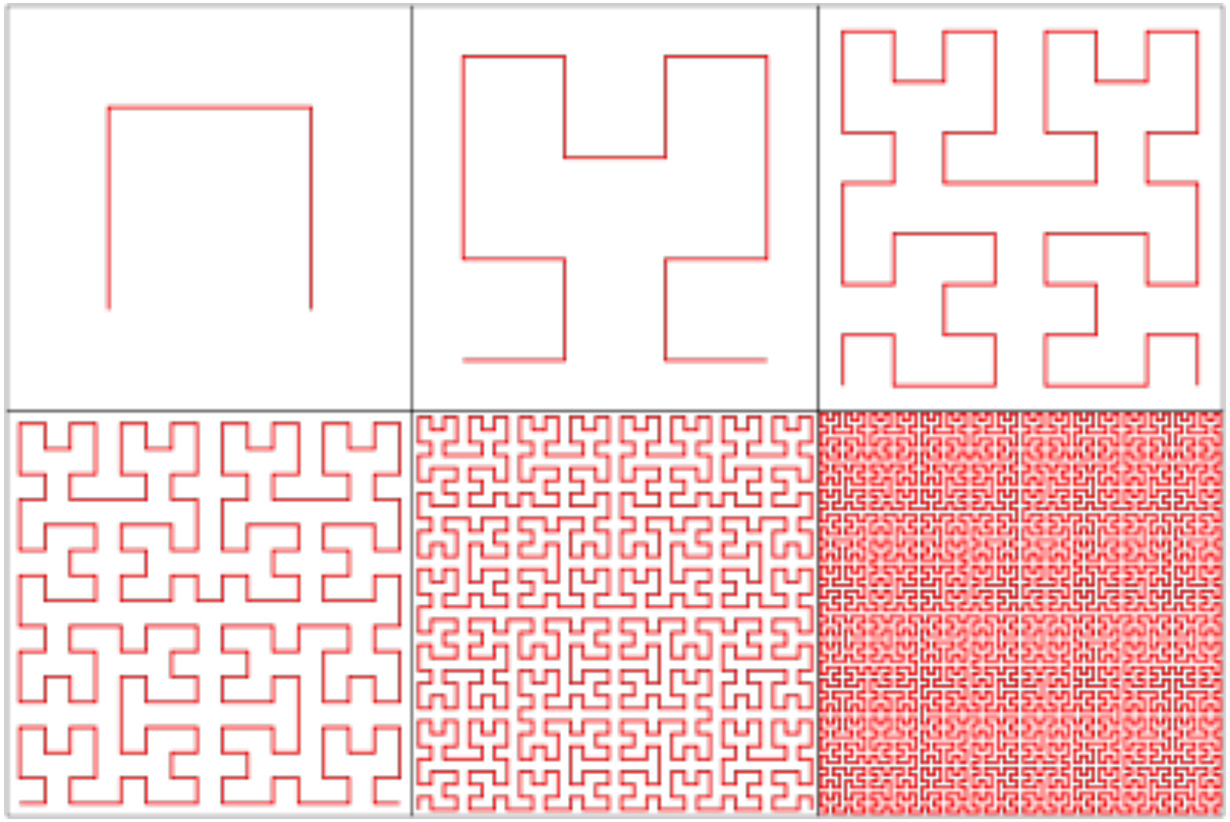
Dimension and measuring

The dimension can tell you a bit about the measurements of a fractal.

- If the dimension is less than 1 then the length and area will be 0.
- If the dimension is between 1 and 2 then length will be infinite but area will be zero.
- If the dimension is bigger than 2 then both length and area will be infinite.

This is how come the Sierpinski triangle has infinite length but zero area!

Even more strange!



Peano curves are able to fill the whole plane!

How long is the coastline of Great Britain?

The coastline of Great Britain is not a self-similar fractal, so we can't use our formula to work out its dimension. However, there are other ways of working out the dimension of fractals.

It can be calculated that the dimension of the coastline of Great Britain is around 1.26.

So how long is the coastline of Great Britain? It is infinitely long!

Next time…

How can we make a flat map of the spherical Earth? Are some methods better than others? Why don't planes fly following straight lines?


Thanks to

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