On the construction of entire functions in the Speiser class

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1. Definitions and preliminary results

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Let $g : \mathbb{C} \to \mathbb{C}$ be quasiregular.

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Corollary

Let $g : \mathbb{C} \to \mathbb{C}$ be quasiregular. Then there exists a quasiconformal map ϕ such that $f := g \circ \phi^{-1}$ is holomorphic.

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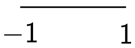

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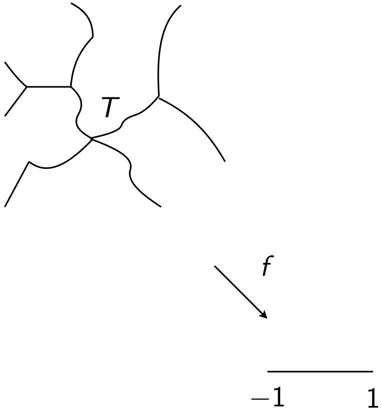

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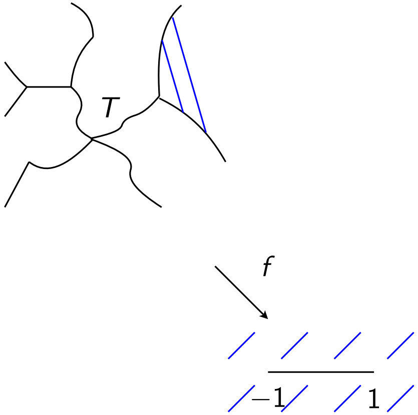

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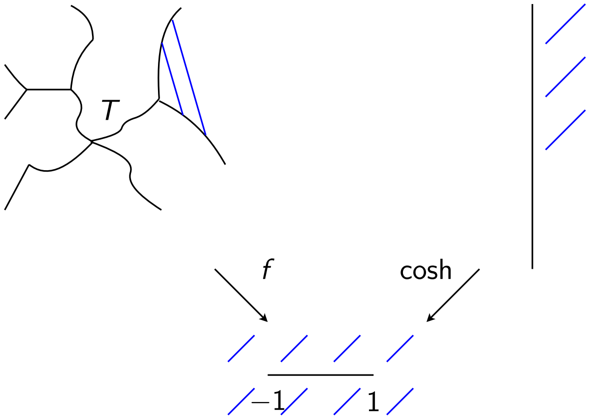

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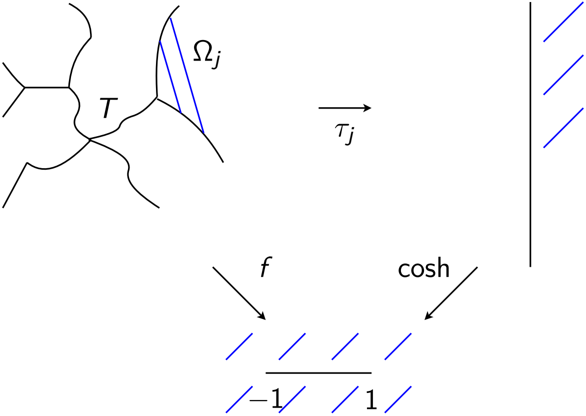

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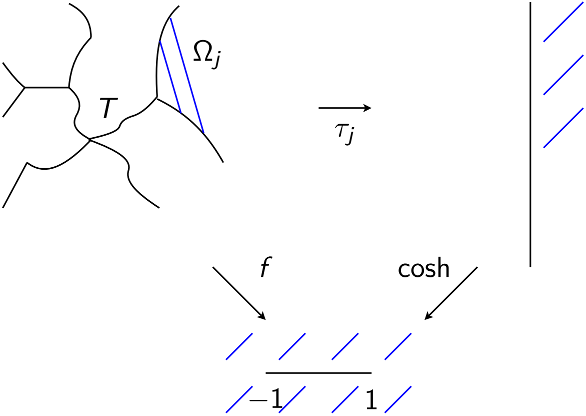

Reverse this procedure!

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- The angles between adjacent edges are uniformly bounded away from zero.
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- For non-adjacent edges e and f, $\frac{\operatorname{diam}(e)}{\operatorname{dist}(e,f)}$ is uniformly bounded.

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- D-component: bounded Jordan domains (they assign other critical values and higher order critical points). We will not use these.

Theorem (Bishop, only L- and R-components)

Suppose T is a bounded geometry tree and suppose τ is conformal from each complementary component of T to its standard version (i.e. left/right half-plane).

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Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = \sigma \circ \tau$ off $T(r_0)$ (a neighbourhood of T). The only singular values of f are ± 1 (critical values coming from the vertices of T) and the singular values assigned by the L components.

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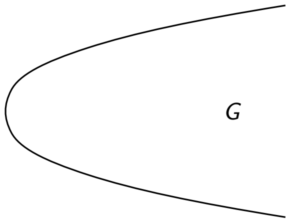
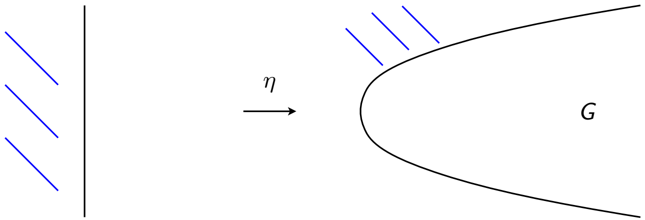
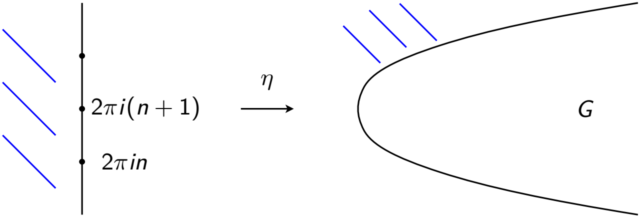
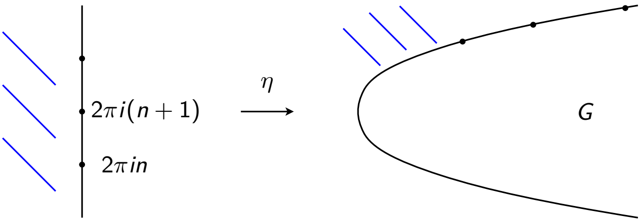
- G symmetric with respect to $\mathbb{R}$ (to make the formulation easier)
- width of tract $\sim$ length of edge (see bounded geometry)


G


η
G



$2\pi i(n+1)$
$2\pi in$
η
G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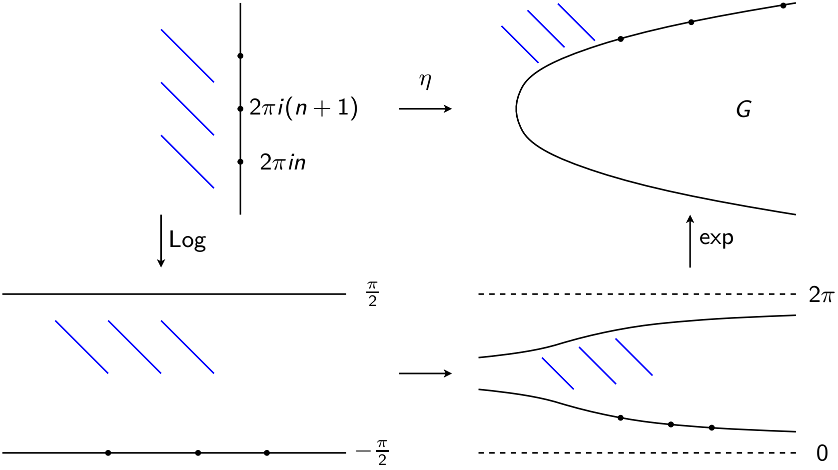
$2\pi i(n+1)$
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Log
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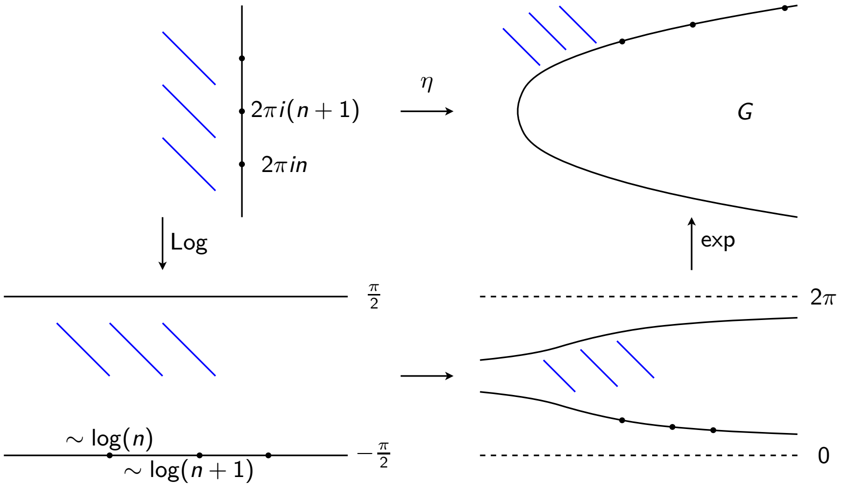

2πi(n + 1)
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Log
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π/2
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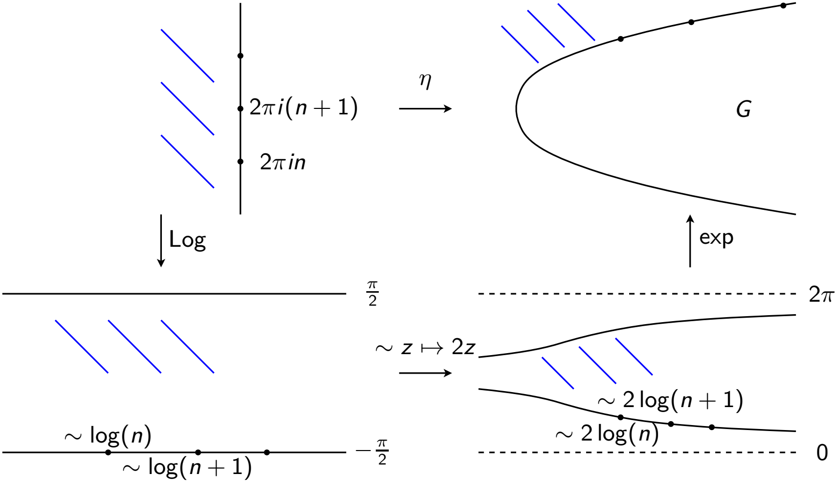
0
∼ log(n)
∼ log(n + 1)



$2\pi i(n+1)$
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G
Log
exp
$\frac{\pi}{2}$
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2π
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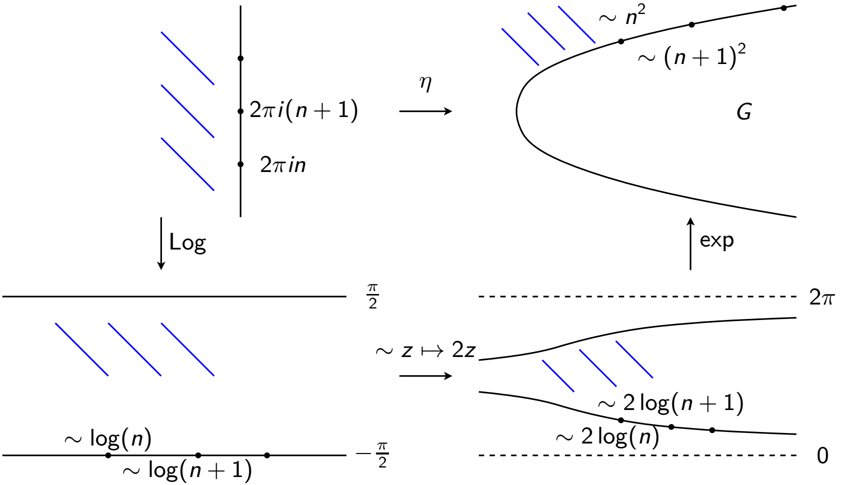

$2\pi i(n+1)$
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η
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$\sim (n+1)^2$
G
Log
exp
$\frac{\pi}{2}$
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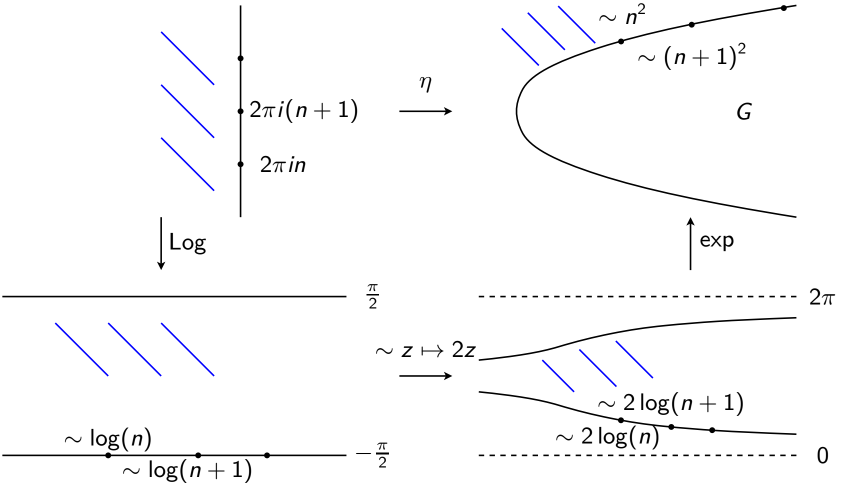

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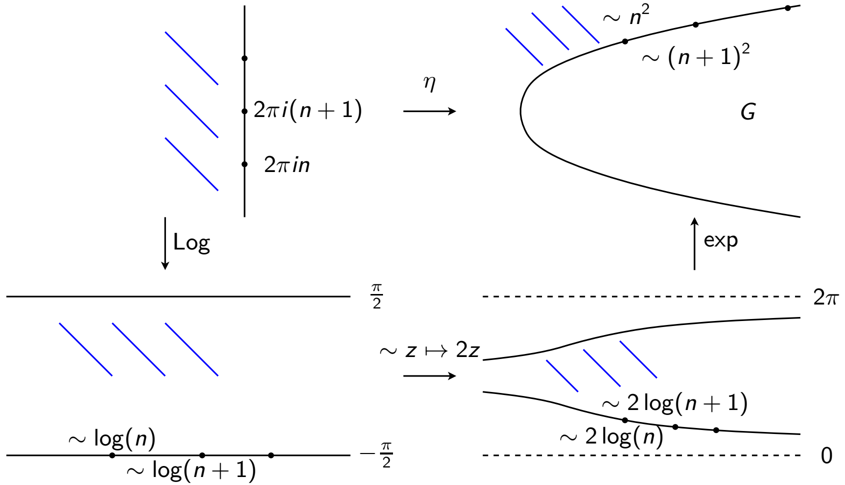

$\Rightarrow$ the length of the n-th edge is $\sim n$ $\Rightarrow$ width of G at $\mathrm{Re} = n^2$ must be n

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Let $0 \le \delta \le \frac{1}{2}$. For $\delta = 0$ let $\gamma < 0$, for $\delta = \frac{1}{2}$ let $\gamma \ge 1$ and let $\gamma \in \mathbb{R}$ otherwise. Let $k > 0$ and

$$G := \left\{ re^{i\varphi} \ : \ r > 1, \, |\varphi| < k \cdot \frac{(\log r)^{\gamma}}{r^{\delta}} \right\}.$$

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Then there exist a quasiregular map g *and constants* $k_0 > 0$*,* $r_0 > 1$ *and* $R \geq 1$

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Remark

ϕ asymptotically conformal: $\lim_{|z| \to \infty} \frac{\phi(z)}{z} = c$ exists, $c \neq 0, \infty$

Thank you very much for your attention.