On boundaries of multiply connected wandering domains

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Outline

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Introduction

Definition (Wandering domain)

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Theorem (Sullivan 1982)

There are no wandering domains for rational functions.

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where $C > 0$ is a small constant, r_1 is large and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers that satisfies the recurrence relation

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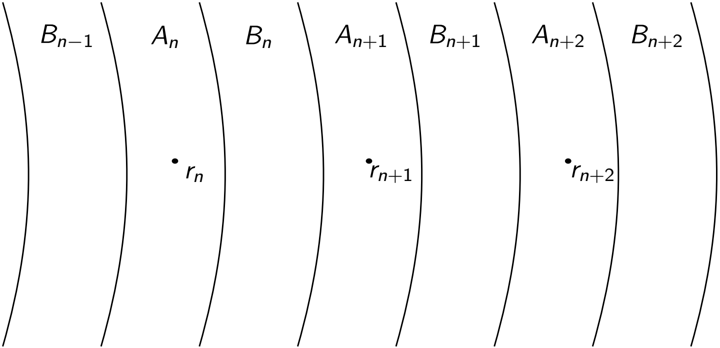
In 1976 Baker was able to show that the U_n are all different and therefore wandering domains.

$$\bullet\, r_n \xrightarrow{\;f\;} \bullet\, r_{n+1} \xrightarrow{\;f\;} \bullet\, r_{n+2}$$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A_n
B_n
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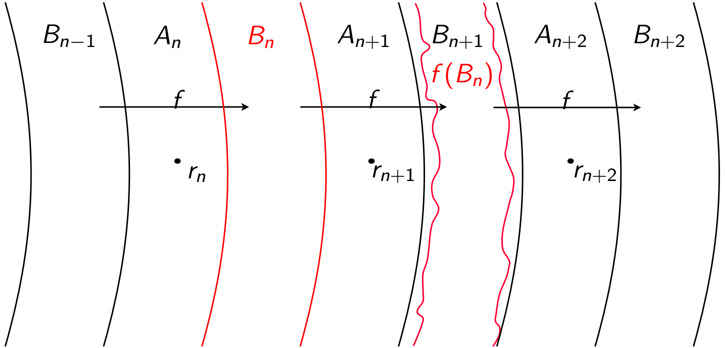

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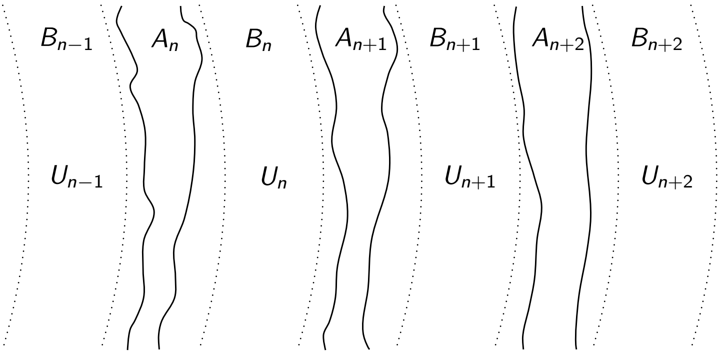

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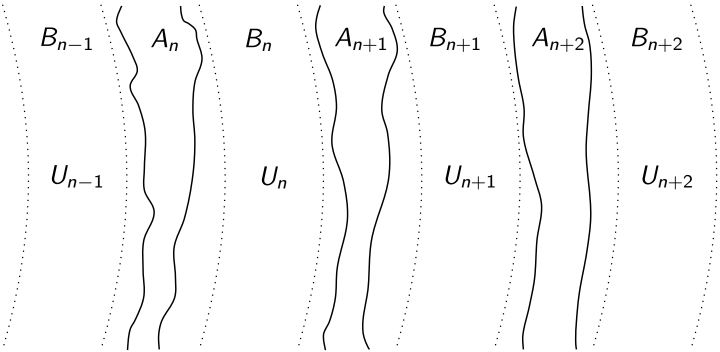

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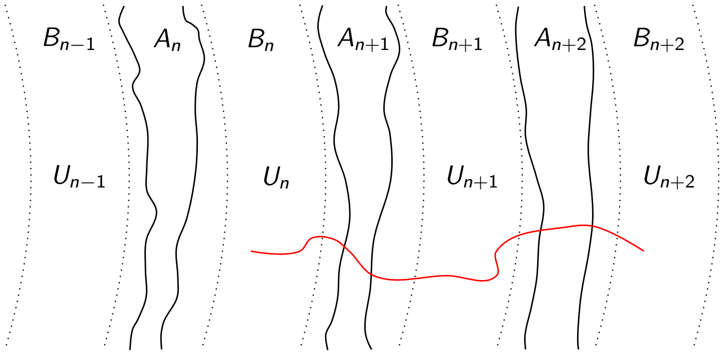

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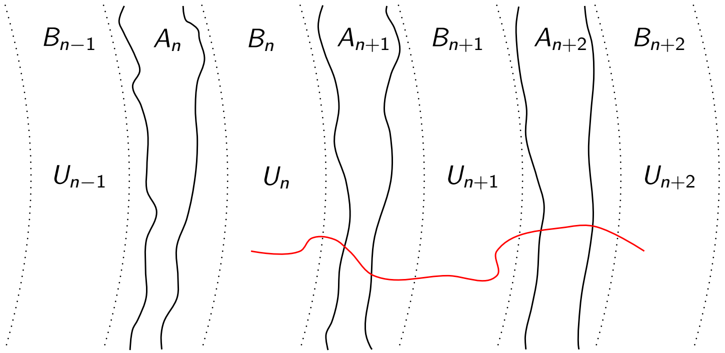

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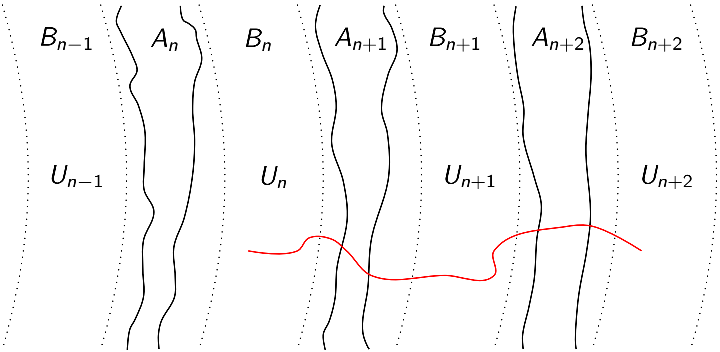

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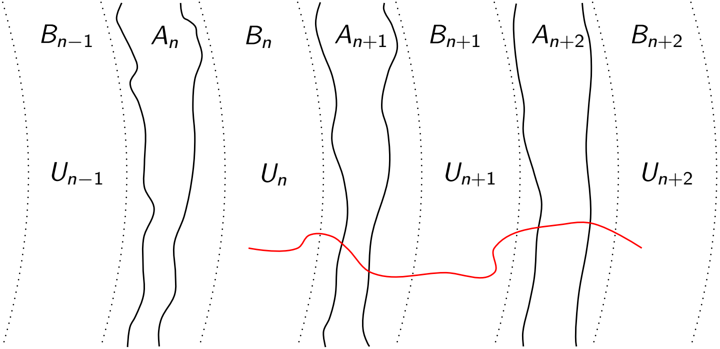

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Baker showed later that every multiply connected wandering domain has similar properties like his first example.

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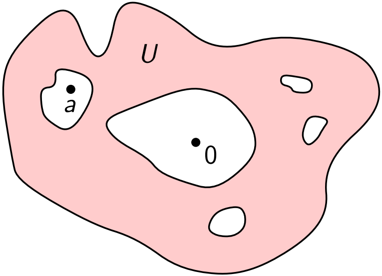
We want to show that under suitable conditions every boundary component of a multiply connected wandering domain is a curve or even a Jordan curve and therefore locally connected.

Results

Definition (Inner and outer boundary)

Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.



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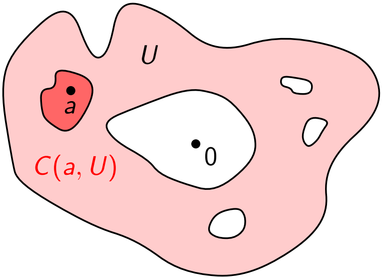

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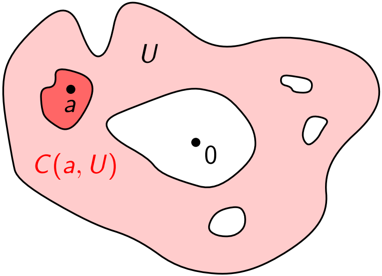

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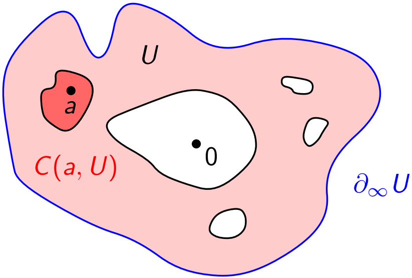

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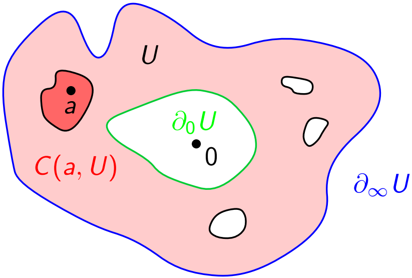

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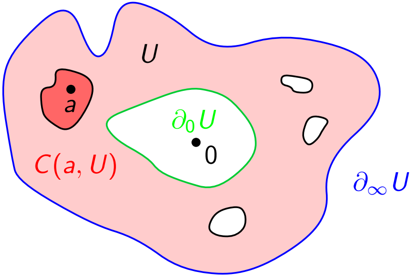
We call $\partial_0 U$ and $\partial_\infty U$ *big boundary components*.



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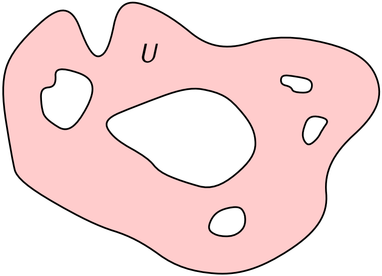

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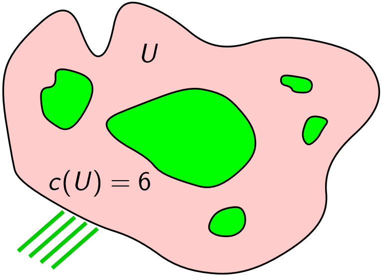

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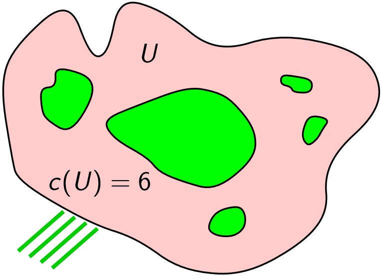

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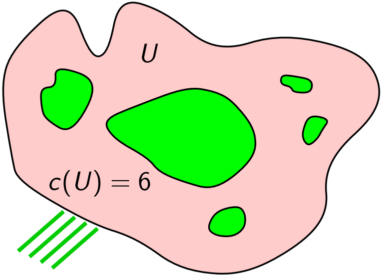

Kisaka and Shishikura showed that the eventual connectivity of a multiply connected wandering domain is either 2 or ∞.

Theorem (Bergweiler, Rippon, Stallard 2013)

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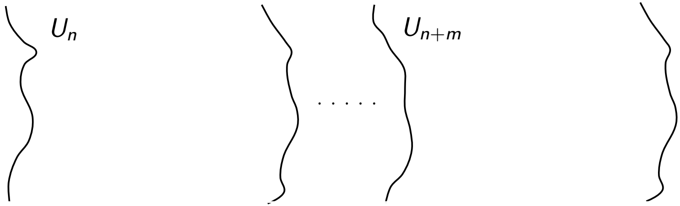

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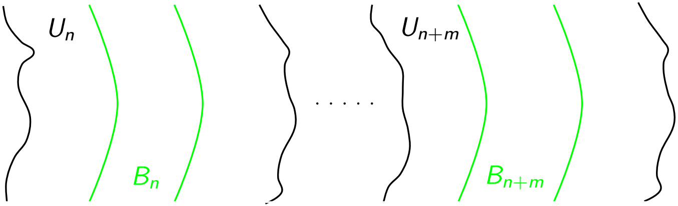

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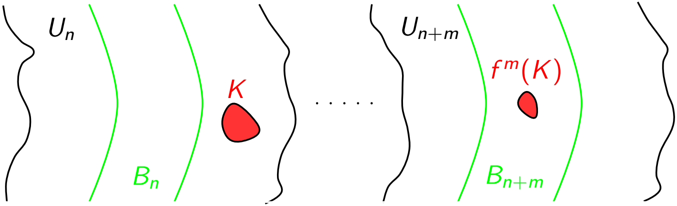

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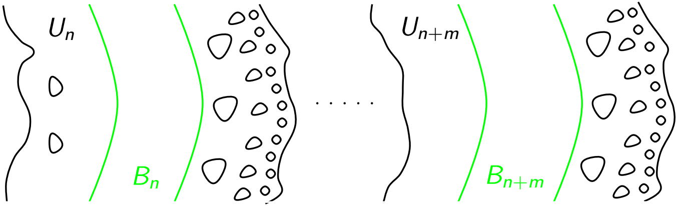

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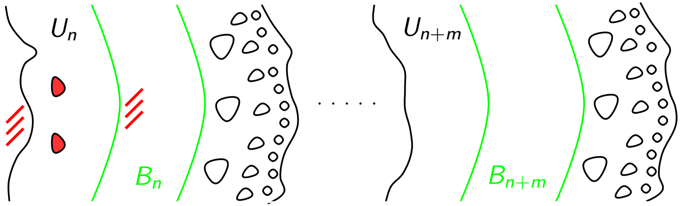

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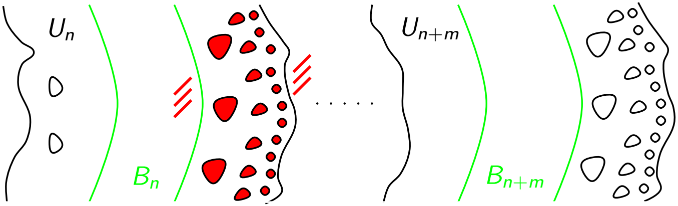

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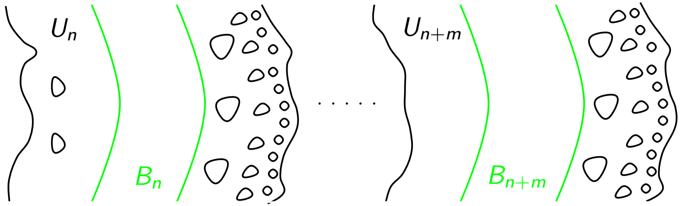

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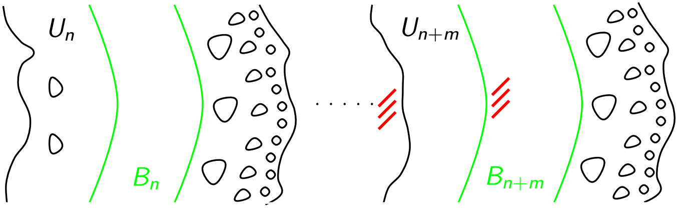

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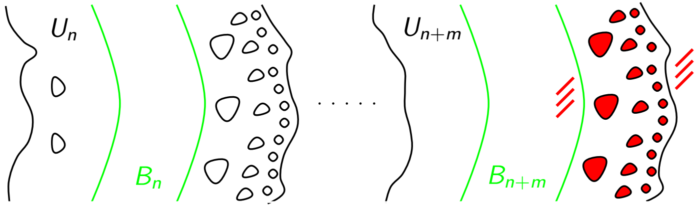

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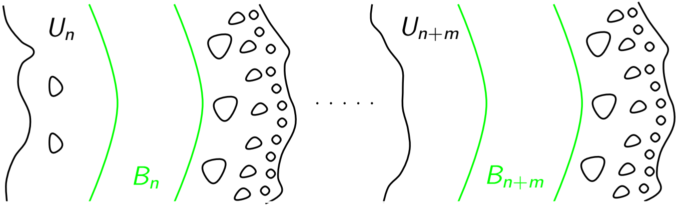

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Theorem 1

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Then all big boundary components are Jordan curves and $\partial_\infty U_{n-1} = \partial_0 U_n$.

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Let Z be an eventually big boundary component of U.
Then Z is a closed (rectifiable) curve. Moreover Z is a (rectifiable) Jordan curve if $f^j(Z)$ does not contain any critical points for all $j \in \mathbb{N}_0$.

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We can apply Theorem 1 and both corollaries for Baker's first example of a wandering domain. This means that every multiply connected wandering domain in Baker's first example is bounded by a countable number of Jordan curves.

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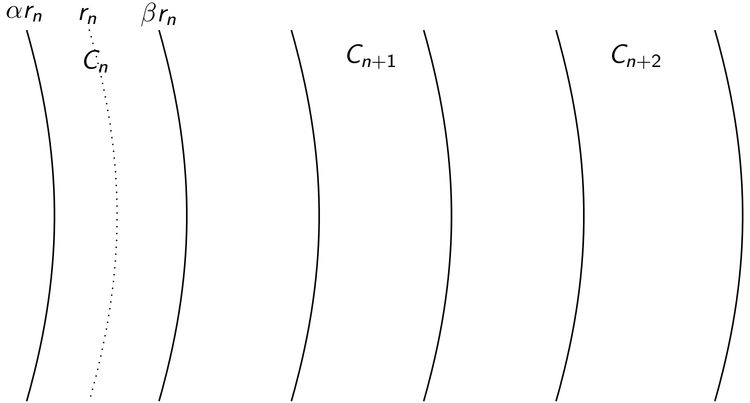

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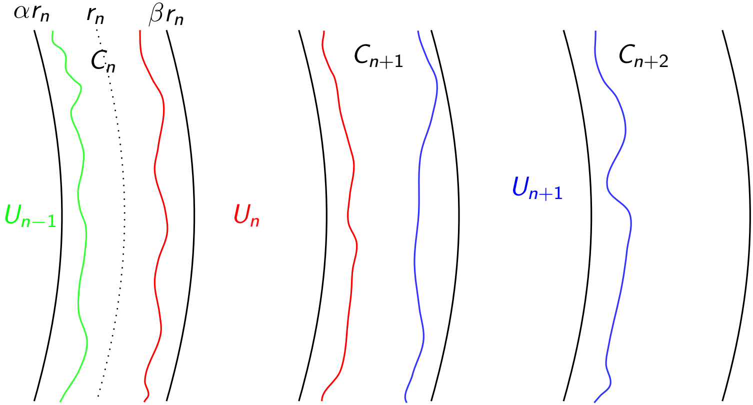

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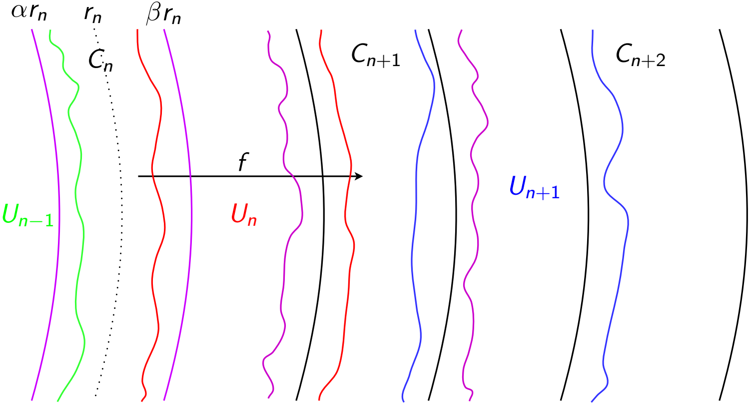

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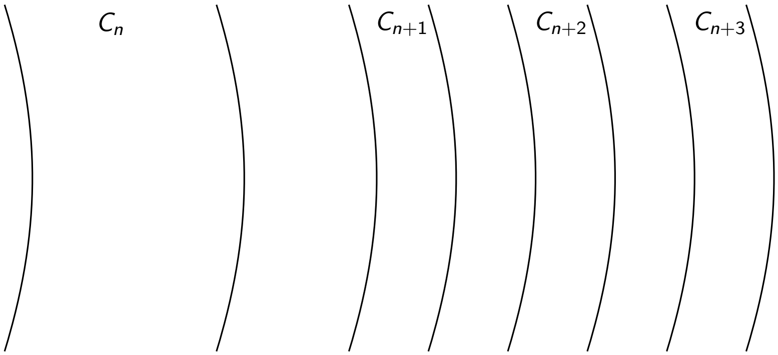

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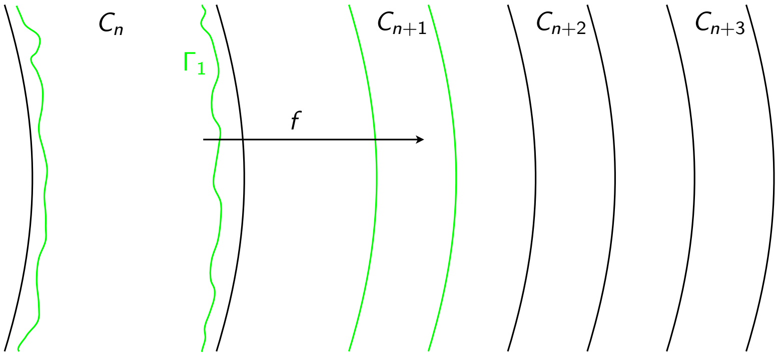

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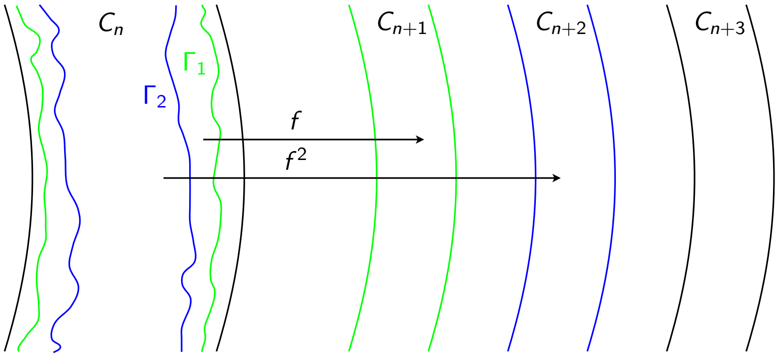

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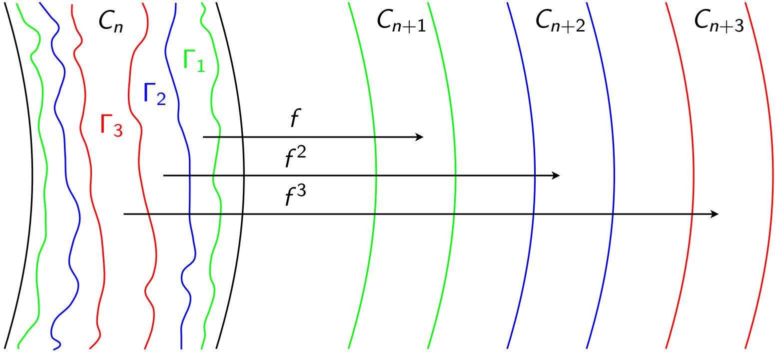

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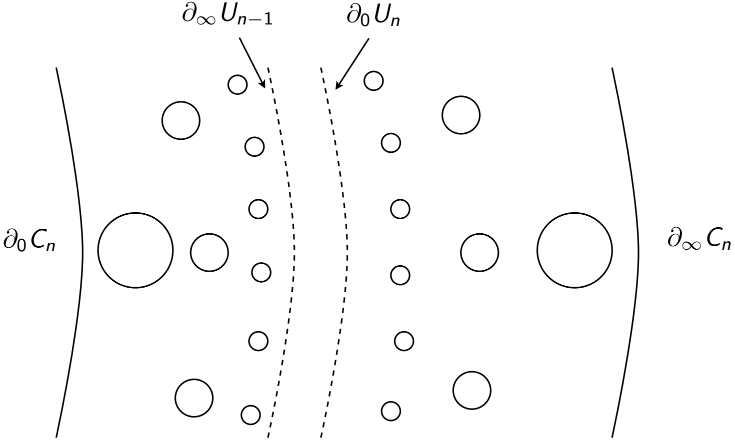
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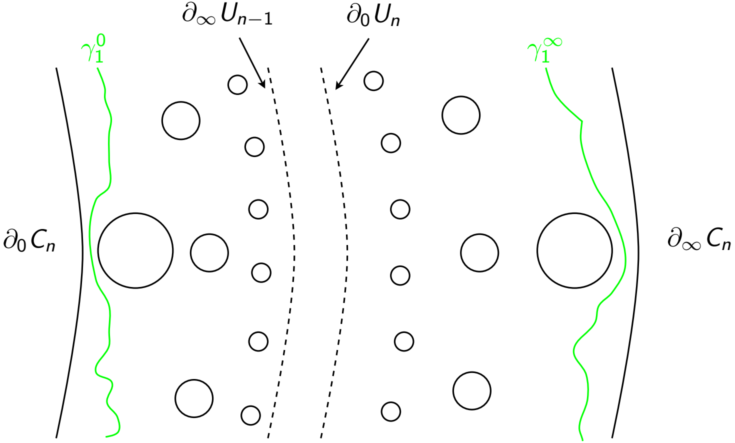
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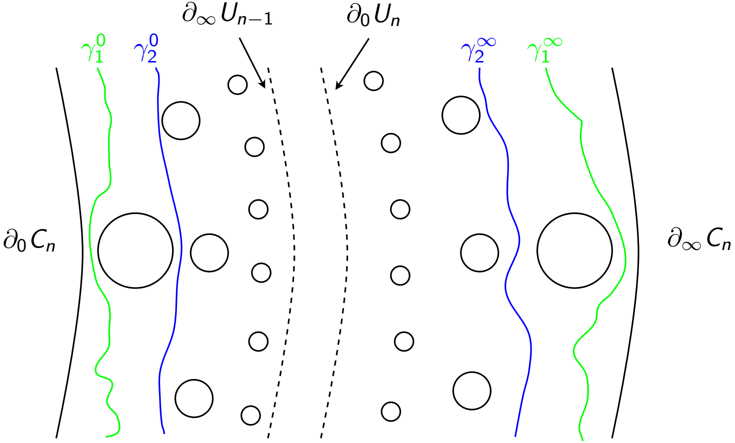
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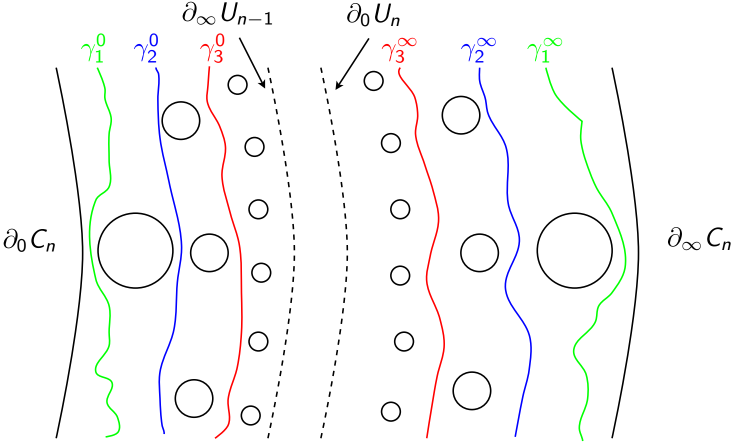

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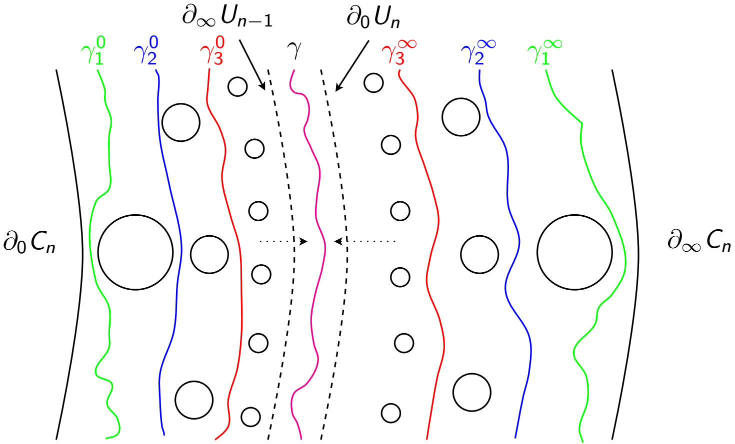

Then we use that f^{-k} is contracting to show that the curves γ_k^0 and γ_k^∞ converge uniformly to the same curve γ with

$$\operatorname{trace}(\gamma) = \bigcap_{k \in \mathbb{N}} \Gamma_k .$$

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In order to prove theorem 2 we exploit that $\left|\arg\left(\frac{z \cdot f'(z)}{f(z)}\right)\right| < \varepsilon_n$ ensures that the curves are only distorted by a very small amount under iteration.

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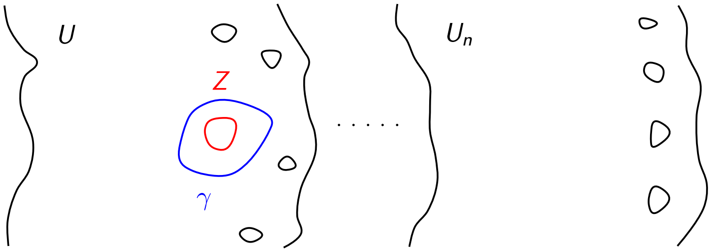

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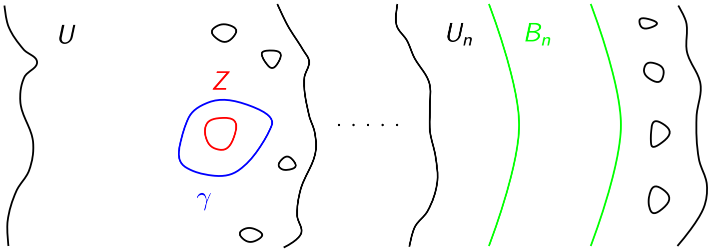

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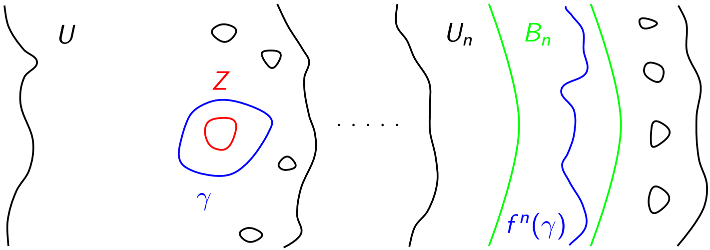

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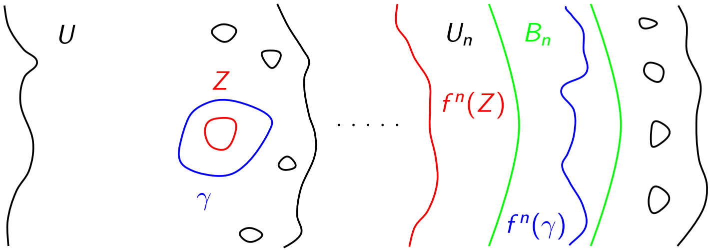

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Bergweiler's and Zheng's example

$$f(z) = C \cdot z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right),$$

where $C > 0$, $k \in \mathbb{N}$ and $(a_n)_{n \in \mathbb{N}}$ is a complex sequence with $|a_n| = r_n$ and $(r_n)_{n \in \mathbb{N}}$ is a fast growing sequence of positive real numbers.

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The sequence $(r_n)_{n \in \mathbb{N}}$ can be chosen such that the conditions of theorem 1 hold. This example includes the first example of Baker. In this case the eventual inner connectivity is 2.

Baker's infinite connectivity example

$$f(z) = C \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right)^k,$$

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The sequence $(r_n)_{n\in\mathbb{N}}$ can be chosen such that the conditions of theorem 1 hold. This example includes the first example of Baker (1984) with a wandering domain with infinite connectivity. In this case the eventual inner connectivity is 2. Bergweiler and Zheng showed that Baker's first example of a wandering domain has also infinite connectivity.

Baker's example of arbitrary order

$$f(z) = C \cdot \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{r_n}\right)^{k_n}\right),$$

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The sequence $(r_n)_{n\in\mathbb{N}}$ can be chosen such that the conditions of theorem 2 hold. This example includes the first example of Baker (1984) with arbitrary order. Bishop's example which was the starting point of my work is also constructed by an infinite product which is similar to the one above.

Thank you for your attention.