HOLOMORPHIC MOTION FOR JULIA SETS OF HOLOMORPHIC FAMILIES OF ENDOMORPHISMS OF $\mathbb{P}^k$

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Abstract. We build measurable holomorphic motions for Julia sets of holomorphic families of endomorphisms of $\mathbb{P}^k$ under various equivalent notions of stability. This generalizes the well-known result obtained by Mane-Sad-Sullivan and Lyubich in dimension 1 and leads to a coherent definition of the bifurcation locus in this setting. Since the usual 1-dimensional techniques no longer apply in higher dimension, our approach is based on ergodic and pluripotential methods. This is a joint work with François Berteloot.

1. The 1-dimensional theory

Let us consider a rational map f on the Riemann sphere $\mathbb{P}^1$. It is possible to decompose $\mathbb{P}^1$ into two completely invariant subsets: the *Fatou set*, that is the open set of points having a neighbourhood there the family of the iterates $\{f^{\circ n}\}$ is normal, and its complement, the *Julia set*. This is where we find the chaotical dynamics. It is a classical result that such a system admits a unique measure of maximal entropy, usually denoted by μ, whose support is precisely the Julia set. Moreover, the Julia set coincides with the closure of the repelling periodic cycles $\mathcal{R}$ for f. Let us now consider a holomorphic family of rational maps, defined in the following way.

Definition 1.1. *A holomorphic family of rational maps is a holomorphic map $f : M\times\mathbb{P}^1 \to M \to \mathbb{P}^1$ of the form $f(\lambda, z) = (\lambda, f_\lambda(z))$, where M is a connected complex manifold and such that all maps $f_\lambda := f(\lambda, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ are rational maps of the same degree $d \geq 2$.*

The problem is the following: we want to study the dependence of the Julia set J_λ for f_λ on the parameter λ. Notice that, as recalled above, since $\overline{\mathcal{R}_\lambda} = J_\lambda$, it is natural to investigate how J_λ varies with λ through the parametrization of the repelling cycles.

In order to do this, we need the following definition.

Definition 1.2. *Let E be a subset of the Riemann sphere and Ω be a complex manifold. Let $\lambda_0 \in \Omega$. A holomorphic motion of E over Ω and centered at λ_0 is a map*

$$h : \Omega \times E \ni (\lambda, z) \mapsto h_\lambda(z) \in \mathbb{P}^1$$

which satisfies the following properties:

i) $h_{\lambda_0} = Id|_E$
ii) $E \ni z \mapsto h_\lambda(z)$ *is one-to-one for every* $\lambda \in \Omega$
iii) $\Omega \ni \lambda \mapsto h_\lambda(z)$ *is holomorphic for every* $z \in E$.

Notice in particular that a holomorphic motion gives a lamination on a subset of the product space $M \times \mathbb{P}^1$. Moreover, let $z_0 \dots f^{n-1}_{\lambda_0}(z_0)$ be an $n-$repelling cycle for f_{λ_0}. By the implicit function theorem, there exists a holomorphic motion of this cycle on a neighbourhood Ω_{λ_0} that conjugates the dynamics ($f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$).

The interest of holomorphic motions (in dimension 1) relies on the fact that any holomorphic motion of a set E extends to the closure of E. This is the content of the so-called λ-lemma and is a quite simple consequence of Picard-Montel theorem:

Lemma 1.3 (λ-lemma, Mané-Sad-Sullivan). *A holomorphic motion h of E extends to a holomorphic motion $\bar{h}$ of $\bar{E}$ (and moreover $\bar{h}$ is continuous).*

As J_λ is the closure of the set of repelling cycles of f_λ, this Lemma implies that the Julia set J_{λ_0} moves holomorphically over a neighbourhood V_{λ_0} of λ_0 in M as soon as *all* repelling cycles of f_{λ_0} move holomorphically on V_{λ_0}. Moreover, the holomorphic motion obtained in this way clearly conjugates the dynamics. This is the content of the following fundamental Theorem.

Theorem 1.4 (Lyubich,Mané-Sad-Sullivan). *Let $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ be a holomorphic family of rational maps. If $\mathcal{R}_\lambda$ moves holomorphically on Ω, then J_λ moves holomorphically on Ω (and the motion conjugates the dynamics).*

We can thus define the set of stable parameters and its complement, the bifurcation locus.

Definition 1.5. *The stable set $\mathcal{S}_{tab} \subset M$ is the set of points in M that have a neighbourhood where $\mathcal{R}_\lambda$ (and thus J_λ) moves holomorphically. Its complement $\mathcal{B}_{if} = \mathcal{S}_{tab}^c$ is the bifurcation locus.*

By definition, $\mathcal{S}_{tab}$ is an open subset of M but it is however not yet clear that it is not empty. It turns out that $\mathcal{S}_{tab}$ is actually dense in M.

Theorem 1.6 (Lyubich,Manè-Sad-Sullivan). *The stable set $\mathcal{S}_{tab}$ is dense in M.*

In order to complete the picture in dimension 1 we need a last definition.

Definition 1.7. *The Lyapounov exponent of the system $(J_\lambda, f_\lambda, \mu_\lambda)$ is*

$$L(\lambda) = \int_{\mathbb{P}^1} \ln |f'_\lambda| d\mu_\lambda.$$

The following theorem by DeMarco gives a fundamental characterization of the bifurcation locus.

Theorem 1.8 (DeMarco). *$dd^c L$ is a positive closed current on M and $\mathcal{B}_{if} \equiv \operatorname{Supp} dd^c L$.*

So, we have the following description.

Theorem 1.9 (Lyubich,Mané-Sad-Sullivan,DeMarco). *Let $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ be a holomorphic family of rational maps. Then, the following are equivalent:*

1. *the repelling cycles move holomorphically;*
2. *the Julia set J_λ move holomorphically;*
3. *$dd^c L = 0$.*

The aim of this work is to generalize this statement in higher dimension, thus giving a coherent definition of the bifurcation locus in this more general setting.

2. The higher-dimensional situation

Let us now move to the general setting. Let M be connected complex manifold of dimension m. A holomorphic family of endomorphisms of $\mathbb{P}^k$ can be seen as a holomorphic mapping

$$f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k \quad , \quad (\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

where the algebraic degree d of f_λ is larger than or equal to 2 and does not depend on λ. For instance, M can be the space $\mathcal{H}_d(\mathbb{P}^k)$ of all degree d holomorphic endomorphisms of $\mathbb{P}^k$, which is a Zariski open subset in some $\mathbb{P}^N$.

It is no longer true that the support of μ contains all the repelling periodic point (Hubbard-Papadopol- Fornaess-Sibony). So, we give the following definition.

Definition 2.1. *The set of J-repelling cycles $\mathcal{R}^J_\lambda := \mathcal{R}_\lambda \cap J_\lambda$ is the set of repelling cycles in J_λ.*

The following Theorem by Briend-Duval assures that $\mathcal{R}^J_\lambda$ still provides a dense subset of J_λ.

Theorem 2.2 (Briend-Duval)**.** *J_λ is the closure of $\mathcal{R}^J_\lambda$.*

Before stating and commenting our result, we sketch the steps of the proof in dimension 1 (1 $\Rightarrow$ 2). We have a point $z \in J_{\lambda_0}$ and we want to extend the holomorphic motion h on the repelling cycle to it. We can approximate z with repelling points by the density of $\mathcal{R}$ in J, and then consider any limit of the motions ρ_j of this repelling points as the motion for z, since the family of the ρ_j's is normal (this is actually true even when $k > 1$). Then, since the ρ_j's have disjoint graphs, Hurwitz Lemma implies that every possible sequence of these motions has the same limit, thus giving the uniqueness for the motion of z.

When $k > 1$, the limits might be not unique and one is therefore led to consider webs instead of laminations. In order to give a precise definition of this, the following framework has been introduced by Berteloot and Dupont. The map f induces a dynamical system $(\mathcal{J}, \mathcal{F})$ where

$$\mathcal{J} := \{\gamma : M \to \mathbb{P}^k \text{: such that } \gamma \text{ is holomorphic and } \gamma(\lambda) \in J_\lambda \text{ for all } \lambda \in M\}$$

$$\mathcal{F}(\gamma)(\lambda) := (f_\lambda \circ \gamma)(\lambda) \forall \lambda \in M$$

Let us denote by $p_\lambda : \mathcal{J} \to \mathbb{P}^k$ the evaluation $p_\lambda(\gamma) := \gamma(\lambda)$.

Definition 2.3. *A* structural web *of f is a probability measure $\mathcal{M}$ with compact support on $\mathcal{J}$ such that*

(1) $\mathcal{F}_\star(\mathcal{M}) = \mathcal{M}$
(2) $p_{\lambda\star}(\mathcal{M}) = \mu_\lambda$ *for every* $\lambda \in M$.

The picture we have to have in mind in order to handle with these objects is the following: we have a set of graphs that a priori can intersect, and what the second condition says is that what we see when we slice at any parameter λ is the equilibrium measure μ_λ.

The strategy is thus the following: to find $\mathcal{M}$ and $\mathcal{S} \subset \mathcal{J}$ such that

$$\begin{cases} \mathcal{M}(\mathcal{S}) = 1 \\ \forall \gamma \neq \gamma' \in \mathcal{S}, \gamma \cap \gamma' = \emptyset \end{cases}$$

In order to state our result we need a last definition.

Definition 2.4. *Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. A* measurable holomorphic motion of the Julia sets J_λ *over M is a subset $\mathcal{L}$ of $\mathcal{J}$ such that*

(1) $\mathcal{F}(\mathcal{L}) = \mathcal{L}$
(2) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ *for every distinct* $\gamma, \gamma' \in \mathcal{L}$
(3) Γ_γ *does not meet the grand orbit of the critical set of f for every* $\gamma \in \mathcal{L}$
(4) $\mu_\lambda\{\gamma(\lambda) : \gamma \in \mathcal{L}\} = 1$ *for every* $\lambda \in M$
(5) *the map* $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *is* d^k*-to-1.*

Theorem 2.5 (Berteloot, B.). *Let $f : M\times\mathbb{P}^k \to M\times\mathbb{P}^k$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$, with M is a simply connected complex manifold. If the J-repelling cycles of f move holomorphically over M there exists a measurable holomorphic motion $\mathcal{L}$ of the Julia sets J_λ over M.*

When M is a simply connected open subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$, we may combine this result with some results of Berteloot-Dupont and get the following theorem.

Theorem 2.6 (Berteloot, B., Dupont). *Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms where M is a simply connected open subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. Then the following assertions are equivalent :*

(A) *$\mathcal{R}^J_\lambda$ moves holomorphically*
(B) *there exists a measurable holomorphic motion $\mathcal{L}$ of the Julia sets J_λ over M.*
(C) *$dd^c L \equiv 0$ on M (here $L(\lambda) = \int_{\mathbb{P}^k} \ln |Df_\lambda| d\mu_\lambda$)*

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