The eventual hyperbolic dimension of entire functions

Joint work with Lasse Rempe-Gillen

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- Can we talk about hyperbolic dynamics near ∞?

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Theorem (Przytycki, Rempe-Gillen)

Let f be an entire function, then

$$\operatorname{hypdim} f = \operatorname{HD} J_{\mathrm{rad}}(f).$$

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- $\mathrm{HD}(J_{\mathrm{rad}}(E_\lambda)) < 2$ (Urbański, Zdunik).

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- The **singular set** of f, $S(f) = \overline{\mathrm{CritVal}(f) \cup \mathrm{Asymp}(f)}$,
- The **postsingular set**, $\mathrm{PS}(f) = \bigcup_{n \geq 0} f^{\circ n}(S(f))$.

The Eremenko-Lyubich class (2)

The singular set is the smallest closed set S for which $f : \mathbb{C} \backslash f^{-1}(S) \to \mathbb{C} \backslash S$ is a covering map.

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- For class $\mathcal{B}$ maps, the singular set do not accumulates on the essential singularity.
- An appropriate definition of hyperbolicity requires to be in $\mathcal{B}$.
- One can use the logarithmic change of coordinate to study class $\mathcal{B}$ functions.

Equivalence of maps

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Let f, g be entire functions. And let C be a class of maps such as "affine", "conformal", "quasiconformal"...
The maps f and g are said to be C-ly equivalent if there exists homeomorphisms $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ in class C such that

$$f \circ \varphi = \psi \circ g.$$

Invariance of escaping dimensions in affine classes

Theorem (Rempe-Gillen, Stallard)

If $f, g \in \mathcal{B}$ are affinely equivalent, then

$$\operatorname{HD} I(f) = \operatorname{HD} I(g).$$

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where $\operatorname{edim} f = \lim_{R\to\infty} \mathrm{HD}\{z \in J : \forall n \geq 0, |f^{\circ n}(z)| \geq R\}$ *is called the eventual dimension.*

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Is it true for quasiconformal equivalence?

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- Both theorem are proved by using the quasiconformal rigidity principle of class $\mathcal{B}$ functions.
- This principle is the extension to transcendental functions of the impossiblity to distinguish two polynomials of the same degree by their dynamics near ∞.

Rigidity of the escaping dynamics (2)

Definition

Let f, g be entire functions. They are **quasiconformally equivalent near infinity** if there exist quasiconformal maps ψ and $\varphi : \mathbb{C} \to \mathbb{C}$, such that $\varphi \circ f(z) = g \circ \psi(z)$ whenever $|f(z)|$ and $|g(\psi(z))|$ are large enough.

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Theorem (Rempe-Gillen)

Let $f, g \in \mathcal{B}$ be quasiconformally equivalent near infinity, then there is a $R \geq 0$, such that they are conjugated on $J_R = \{z : \forall n, |f^{\circ n}(z)| \geq R\}$ by a quasiconformal map of $\mathbb{C}$.

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Theorem (Rempe-Gillen)

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The eventual hyperbolic dimension

Definition

Let f be a transcendental entire function, the **eventual hyperbolic dimension** of f is defined as:

$$\operatorname{evhypdim} f = \limsup_{R\to\infty} \{\operatorname{HD} X : X \subset \{|z| > R\} \text{ is hyperbolic for } f\}$$

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This is also a consequence of the rigidity principle:

- Affine equivalence classes yields analytic families of quasiconformally equivalent maps.
- The semiconjugacy relates hyperbolic sets of f and g close to ∞ with arbitrary small dilatation.

Poincaré functions (1)

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  where p is the period and ρ the multiplier, and L sends 0 on one of the points of the periodic cycle.
- We choose the unique Poincaré function such that $L'(0) = 1$.

Poincaré functions (2)

Example

Let $f(z) = z^d$, with $d \geq 2$. Then the normalised Poincaré function associated to the fixed point $z = 1$ is $L(z) = e^z$.

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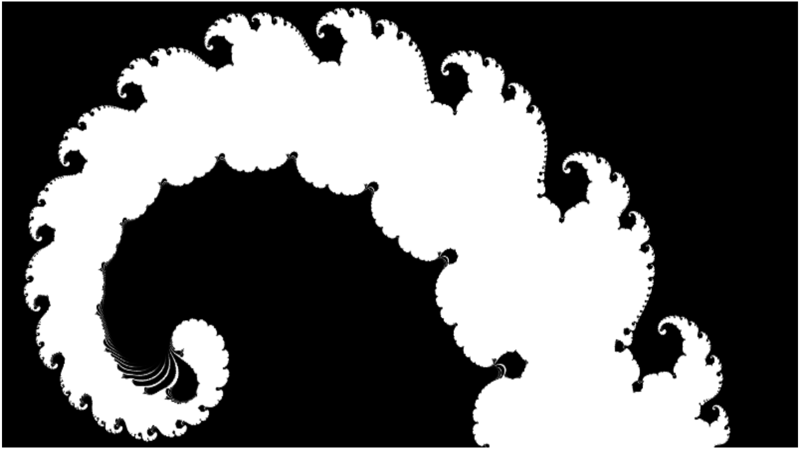
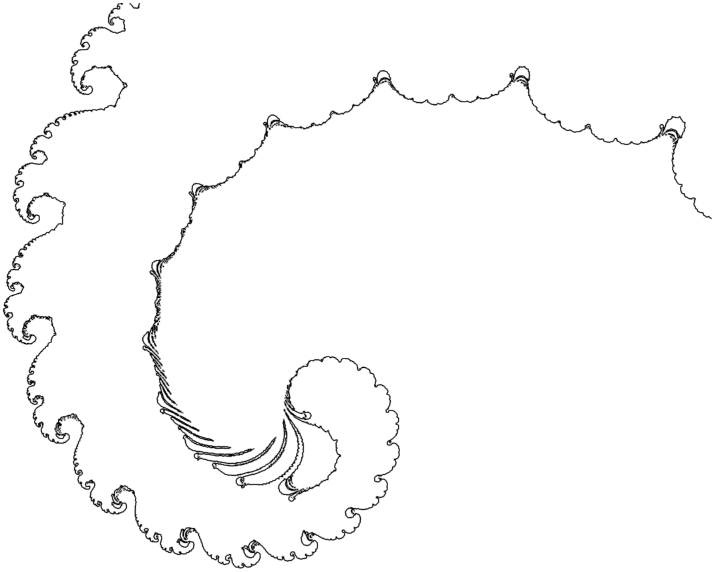
- In particular, if the map f is hyperbolic, the function L belongs to the class $\mathcal{B}$.





Theorem

Let P be a hyperbolic polynomial with connected Julia set, consider a repelling fixed point of P and let L be the Poincaré function of that fixed point. Then,

$$\operatorname{evhypdim} L = \operatorname{HD} J(P),$$

where $J(P)$ is the Julia set of P.

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Remark

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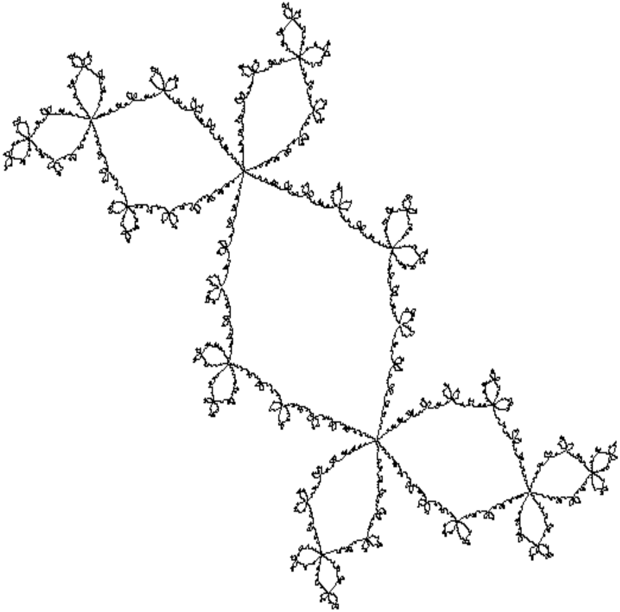
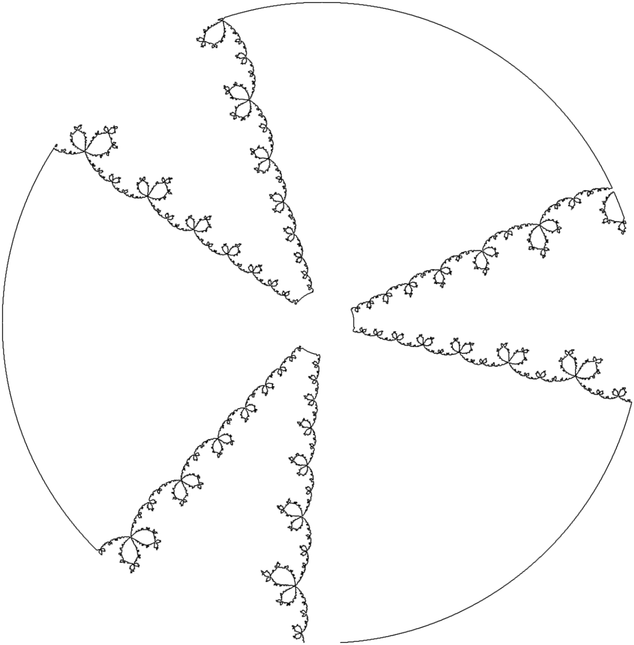
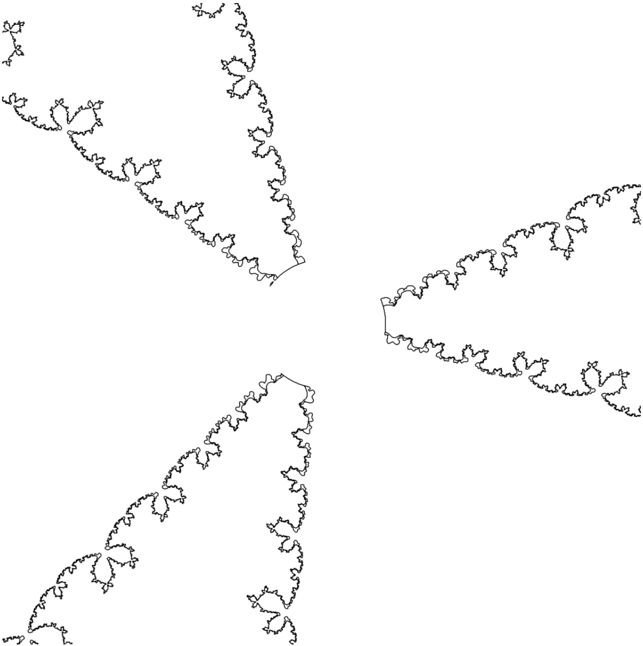
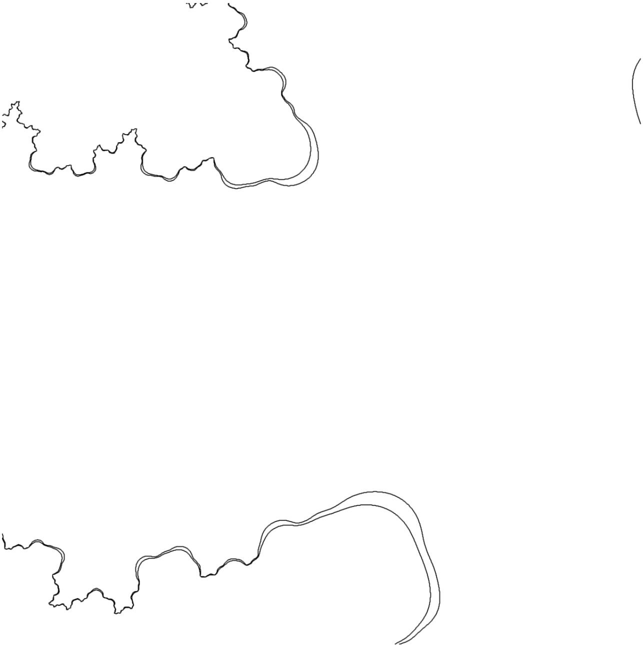
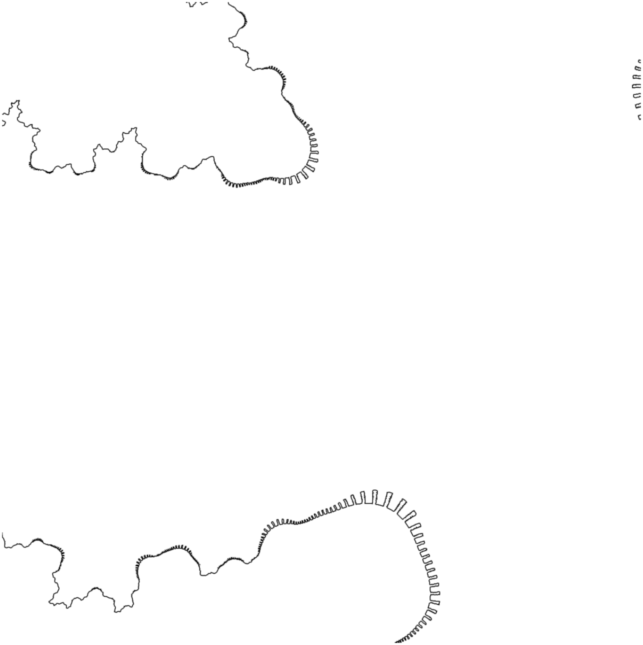
Proof uses thermodynamics formalism and transfer of properties from the P-dynamical plane to the L-dynamical plane by the semiconjugacy.











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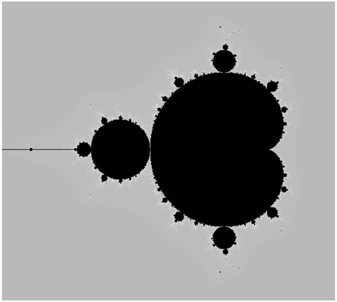

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Corollary

The eventual hyperbolic dimension is not invariant by quasiconformal conjugacy.

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Question

What about the eventual dimension? The dimension of the escaping set?

THANK YOU !