A Newhouse phenomenon in transcendental dynamics

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Attractors

We will consider entire maps $f : \mathbb{C} \to \mathbb{C}$ as dynamical systems.

Recall that a period p cycle $\langle\zeta\rangle$ is said to be *attracting, indifferent* or *repelling* according as the *multiplier* $(f^p)'(\zeta)$ is less than, equal to, or greater than 1. The multiplier is 0 precisely when the cycle contains a critical point : such a cycle is said to be *superattracting*.

- A polynomial $f : \mathbb{C} \to \mathbb{C}$ has only finitely many attractors [Fatou]. In fact, a polynomial of degree D has at most $D - 1$.
- A transcendental $f : \mathbb{C} \to \mathbb{C}$ may have infinitely many attractors. For example, $z \mapsto z - \sin z$ has infinitely many superattractors.

Singular Values

For entire $f : \mathbb{C} \to \mathbb{C}$, we denote by :

- $\Gamma(f)$ the set $\{z : f'(z) = 0\}$ of all *critical points*,
- $C(f)$ the set $f(\Gamma(f))$ of all *critical values*,
- $A(f)$ is the set of all finite *asymptotic values* (limits along paths tending to infinity),
- $S(f)$ the set of all finite values which are *singular* in the sense of covering space theory : $S(f) = \overline{C(f) \cup A(f)}$.
- $\Pi(f)$ the set of finite values which are attained only finitely often. By Picard's Theorem, $\#\Pi(f) \leq 1$ for any entire transcendental f.

Finite and Bounded type maps

We say that $f : \mathbb{C} \to \mathbb{C}$ is of

- *finite type* if $S(f)$ is finite,
- *bounded type* if $S(f)$ is bounded.

Both conditions are preserved under composition, hence by iteration, since $S(f \circ g) = \overline{g(S(f))} \cup S(g)$ for any entire maps f and g.

Theorem (Eremenko-Lyubich)

A finite type transcendental $f : \mathbb{C} \to \mathbb{C}$ has only finitely many attractors. In fact, at most $\#S(f)$ many.

Question (Mihaljević-Brandt)

What about bounded type transcendental maps ?

Results

Theorem

There exists a bounded type entire map with infinitely many attractors.

In fact, bounded type entire maps with infinitely many attractors are prevalent in suitable families. The following is analogous to the *Newhouse phenomenon* of higher dimensional dynamics :

Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map such that the interior of the closure of the critical value set contains a repelling fixed point. There exist a neighborhood of 0 and a residual subset $\mathfrak{R}$ such that for any $\beta \in \mathfrak{R}$ the map $f + \beta$ has infinitely many attractors.

Bifurcation

For entire $f : \mathbb{C} \to \mathbb{C}$, we denote by $E(f)$ the set of all $z \in \mathbb{C}$ with finite backward orbit $\bigcup_{k=0}^{\infty} f^{-k}(z)$.

- There is at most one point in $E(f)$. Indeed, if f is transcendental than $E(f) \subseteq \Pi(f)$, by Picard's Theorem.
- The backward orbit of any other point accumulates everywhere on the Julia set of f.

Bifurcation

Proposition

Let $\lambda \mapsto f_\lambda$ be an analytic family of entire maps parametrized by a connected open neighborhood Λ of 0 in $\mathbb{C}$, and let $\lambda \mapsto \chi_\lambda$ and $\lambda \mapsto \zeta_\lambda$ be analytic functions defined on Λ. Assume that :

- *for every $\lambda \in \Lambda$, the point ζ_λ is a repelling fixed point of f_λ,*
- *for every $\lambda \in \Lambda$, the point χ_λ is a critical point of f_λ,*
- *the function $\lambda \mapsto \zeta_\lambda - f_\lambda(\chi_\lambda)$ vanishes at 0 but not identically,*
- $\chi_0 \notin E(f_0)$.

Then for any sufficiently large positive integer p, there exists $\mu \in \Lambda$ such that χ_μ has period p under f_μ.

Deformation

Theorem

Let f be an entire map with repelling fixed point ζ, let $D \ni \zeta$ be a disc with $\overline{D} \cap S(f) \subseteq \{\zeta\}$, and let $\mathcal{K}$ be the set of all connected components of $f^{-1}(D)$. Consider the set $\mathfrak{B}$ of all functions $V \mapsto \mathbf{b}_V$ from $\Upsilon = \{V \in \mathcal{K} : d_V > 1\}$ to D whose image is bounded in D. Note that $\mathfrak{B}$ is an open neighborhood of the origin in the Banach space $\ell^\infty(\Upsilon)$. There exists an analytic family $\mathbf{b} \mapsto f_{\mathbf{b}}$ such that for any $\mathbf{b} \in \mathfrak{B}$:

- *$f_{\mathbf{b}} \circ \psi^{-1}$ agrees with f outside $f^{-1}(D)$,*
- *$f_{\mathbf{b}}$ restricts to a cover $\psi(V \setminus f^{-1}(0)) \to D \setminus \{\mathbf{b}_V\}$ for each $V \in \Upsilon$.*

Moreover, the family $\mathbf{b} \mapsto f_{\mathbf{b}}$ has the following properties :

- $C(f_{\mathbf{b}}) = \{\mathbf{b}_V : d_V < \infty\} \cup (C(f) \setminus \{\zeta\})$,
- $A(f_{\mathbf{b}}) = \{\mathbf{b}_V : d_V = \infty\} \cup (A(f) \setminus \{\zeta\})$,
- $\Pi(f) = \varnothing$ *implies* $\Pi(f_{\mathbf{b}}) = \varnothing$.

Order

Recall that the *order* of f is

$$\rho(f) = \limsup_{R\to\infty} \frac{\log_+ \log_+ \sup_{|z|=R} |f(z)|}{\log R}$$

where $\log_+ R = \max(0, \log R)$.

By the Ahlfors Distortion Theorem,

- $\rho(f) \geq \frac{1}{2}$ for any bounded type transcendental map,
- $\rho(f) \geq 1$ for any finite type map with $A(f) \neq \varnothing$,
- if $\rho(f) < \infty$ then f is of bounded type precisely when $C(f)$ is bounded.

Order

For the family $\mathbf{b} \mapsto f_{\mathbf{b}}$, we have

$$\rho(f_{\mathbf{b}}) = \rho(f)$$

provided that :

- $\{V : |\mathbf{b}_V| > \epsilon\}$ is finite for every $\epsilon > 0$ and $\mathbf{b}_V = 0$ whenever $d_V = \infty$,

or

- if f has the *Area Property* :

$$\int_{f^{-1}(K)\setminus\mathbb{D}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} < \infty$$

for every compact set $K \subset \mathbb{C} \setminus S(f)$.

Lemma

Consider the entire map $\mathfrak{f} : \mathbb{C} \to \mathbb{C}$ *given by*

$$\mathfrak{f}(z) = \left(\sin \tfrac{\pi}{2}\sqrt{z}\right)^2 = \frac{1 - \cos \pi\sqrt{z}}{2}.$$

1. $\mathfrak{f}$ *has a repelling fixed point at* 0 *with multiplier* $\frac{\pi^2}{4}$.
2. $\Gamma(f) = \{n^2 : n \in \mathbb{Z}\} \setminus \{0\}$ *consists of simple critical points. The corresponding critical values* $\mathfrak{f}(n^2)$ *are* 1 *for odd* n *and* 0 *for even* n*. Moreover,* $\mathfrak{f}^{-1}(1) \subset \Gamma(f)$ *and* $\mathfrak{f}^{-1}(0) \setminus \{0\} \subset \Gamma(f)$.
3. $A(f) = \varnothing$.
4. f *is a map of finite type.*
5. $\Pi(f) = \varnothing$.
6. $\rho(\mathfrak{f}) = \frac{1}{2}$.
7. *The map has the Area Property.*

Theorem

There exists a bounded type entire map g with infinitely many attractors, and such that

- $\rho(g) = \frac{1}{2}$,
- $A(g) = \varnothing$,
- $C(g)$ *has a unique accumulation point, or the closure of* $C(g)$ *has nonempty interior [whichever is desired].*