Semigroups of Möbius transformations

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- Joint work with Ian Short -

The Open University

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Möbius transformations

Möbius transformations are the *conformal automorphisms* of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

That is the bijective functions on $\widehat{\mathbb{C}}$ which preserve angles and their orientation.

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By decomposing the action of any given Möbius transformation into a composition of inversions in spheres orthogonal to $\mathbb{S}^2$, the action of $\mathcal{M}$ may be extended to a conformal action on $\mathbb{R}^3 \cup \{\infty\}$.

In particular $\mathcal{M}$ gives a conformal action on the closed unit ball, which it preserves.

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The hyperbolic metric, $\rho(\cdot\ ,\cdot)$ on $\mathbb{B}^3$

The hyperbolic metric ρ on $\mathbb{B}^3$ is induced by the infinitesimal metric

$$ds = \frac{|d\mathbf{x}|}{1 - |\mathbf{x}|^2}.$$

- From any point inside $\mathbb{B}^3$ the distance to the ideal boundary, $\mathbb{S}^2$, is infinite.
- Geodesics are circular arcs which when extended land orthogonally on $\mathbb{S}^2$.

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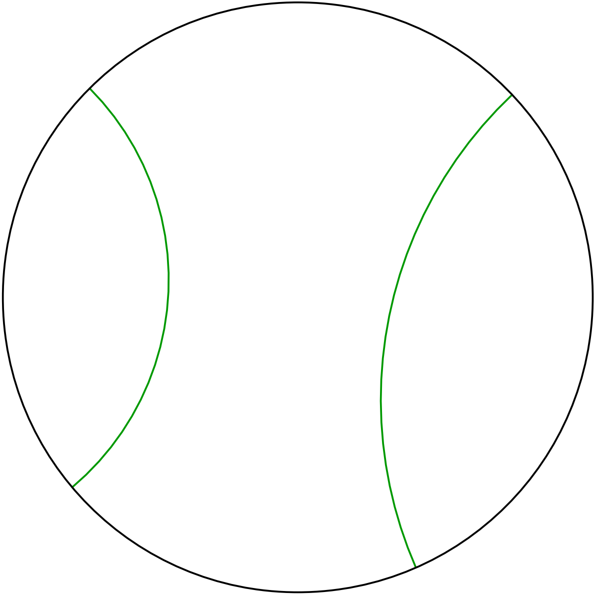

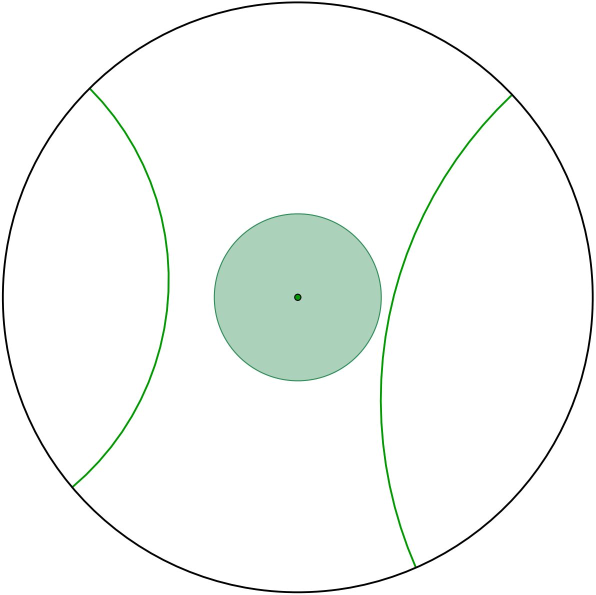

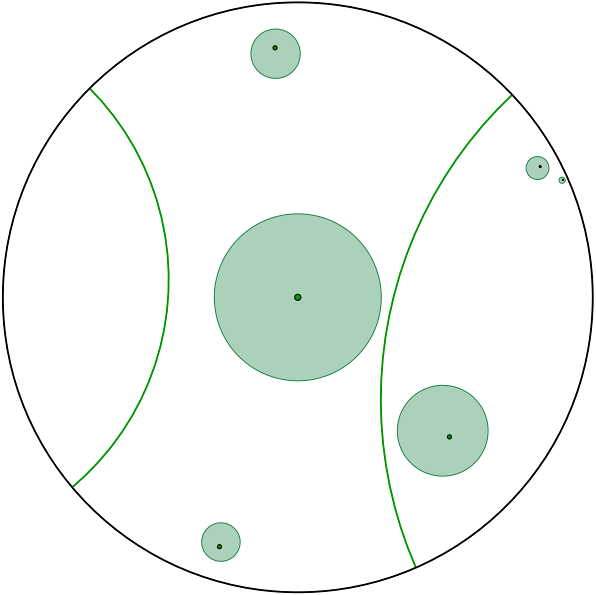

Aside from the identity, there are three types of Möbius transformation.

- Loxodromic transformations
  Conjugate to $z \longmapsto \lambda z$ where $|\lambda| \neq 1$.
  Have two fixed points, one attracting and one repelling.

- Elliptic transformations
  Conjugate to $z \longmapsto \lambda z$ where $|\lambda| = 1$.
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- Parabolic transformations
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Semigroups

Definition

Given a set $\mathcal{F}$ of Möbius transformations, the *semigroup* S generated by $\mathcal{F}$ is the set of finite (and non-empty) compositions of elements from $\mathcal{F}$.

We write $S = \langle \mathcal{F} \rangle$ as the semigroup generated by $\mathcal{F}$.

Limit sets

Let S be a semigroup of Möbius transformations.

Definition

The forwards limit set of S is the set

$$\Lambda^+(S) = \left\{ z \in \mathbb{S}^2 \mid \lim_{n\to\infty} g_n(\zeta) = z \text{ for some sequence } g_n \text{ in } S \right\}.$$

Similarly the backwards limit set of S is given by

$$\Lambda^-(S) = \left\{ z \in \mathbb{S}^2 \mid \lim_{n\to\infty} g_n^{-1}(\zeta) = z \text{ for some sequence } g_n \text{ in } S \right\}.$$

Since each g_n is an isometry of the hyperbolic metric, these definitions are independent of the choice of $\zeta \in \mathbb{B}^3$.

Three characterisations

Write

$$J(S) = \text{subset of } \mathbb{S}^2 \text{ upon which S is not a } \textit{normal family}.$$

Theorem D. Fried, S. Marotta and R. Stankewitz (2012)

For except for certain "Elementary" semigroups,

$$\begin{aligned} \Lambda^-(S) &= J(S) &= \overline{\{\text{Repelling fixed points of } S\}} \\ \Lambda^+(S) &= J(S^{-1}) &= \overline{\{\text{Attracting fixed points of } S\}}. \end{aligned}$$

Properties (Fried, Marotta and Stankewitz)

- Both Λ^+, Λ^- are closed.
- Either $|\Lambda^+| < 3$ or Λ^+ is a perfect set. Similarly for Λ^-.
- Λ^+ is *forward invariant under S*, that is $g(\Lambda^+) \subseteq \Lambda^+$ for all $g \in S$.
- If Λ^+ contains at least two points then it is the smallest closed forwards invariant set containing at least two points.
- Λ^- is *backwards invariant under S*, that is $g^{-1}(\Lambda^-) \subseteq \Lambda^-$ for all $g \in S$.
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Elementary semigroups

$\mathcal{F} = \left\{ z \longmapsto e^{i\theta} z \right\}$
$\Lambda^- = \Lambda^+ = \emptyset.$

$\mathcal{F} = \{ z \longmapsto 2z \}$
$\Lambda^- = \{0\}, \quad \Lambda^+ = \{\infty\}.$

$\mathcal{F} = \{ z \longmapsto z + 1 \}$
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$\mathcal{F} = \left\{ z \longmapsto \frac{1}{3}z,\ z \longmapsto \frac{1}{3}z + \frac{2}{3} \right\}$
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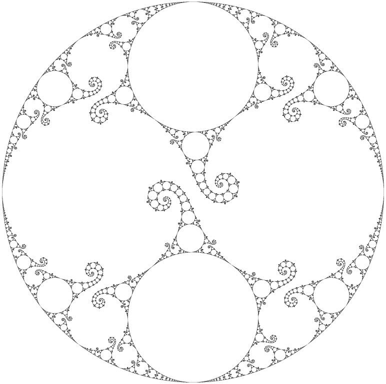

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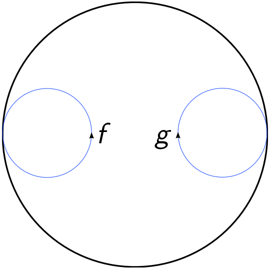

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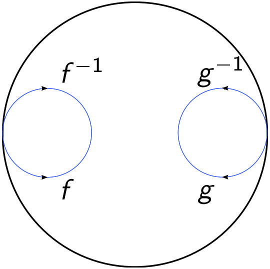

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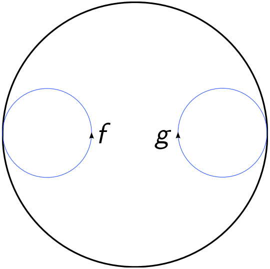

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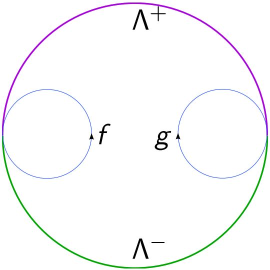

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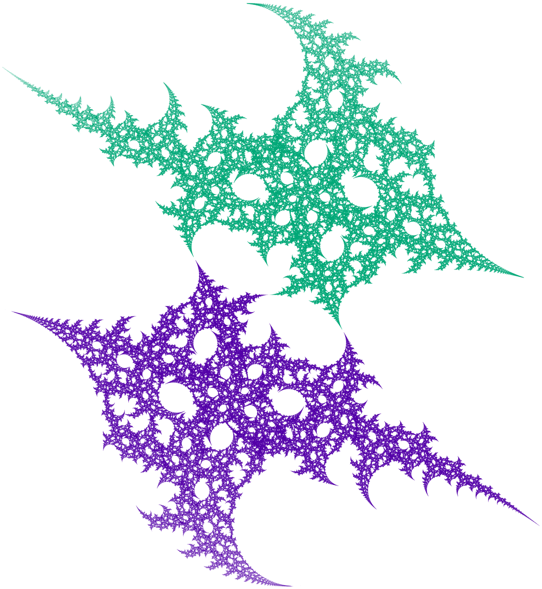
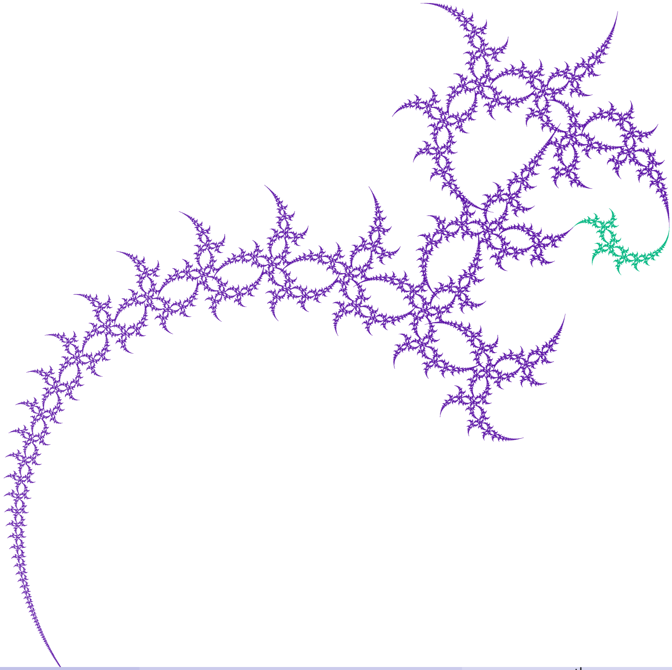
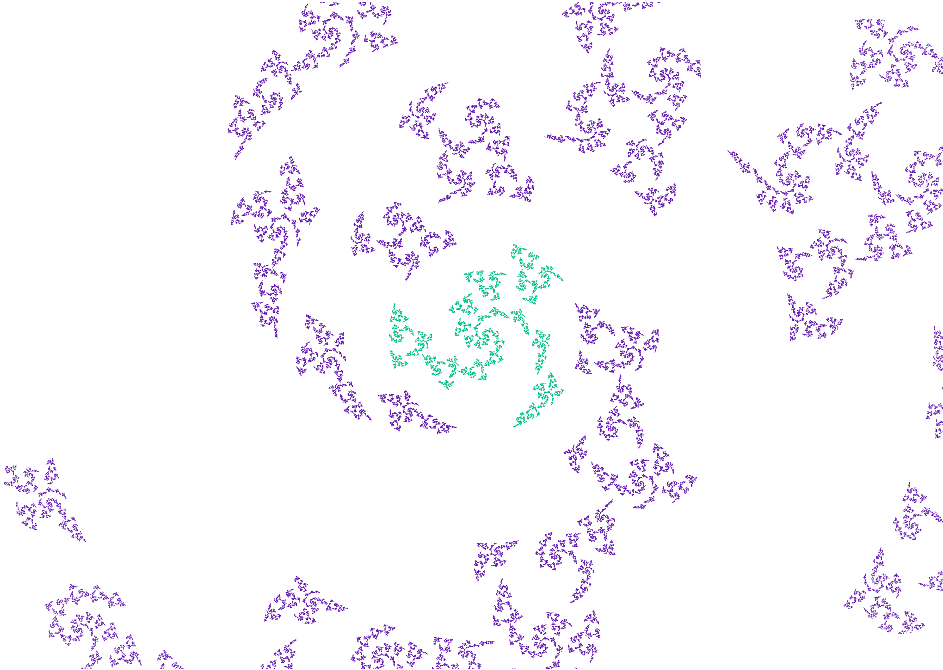
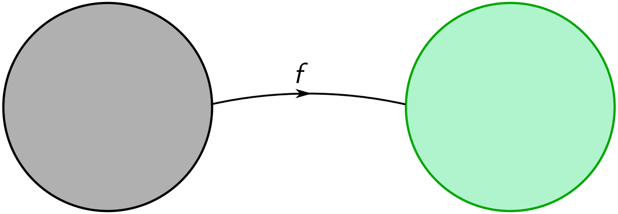






Schottky Semigroups



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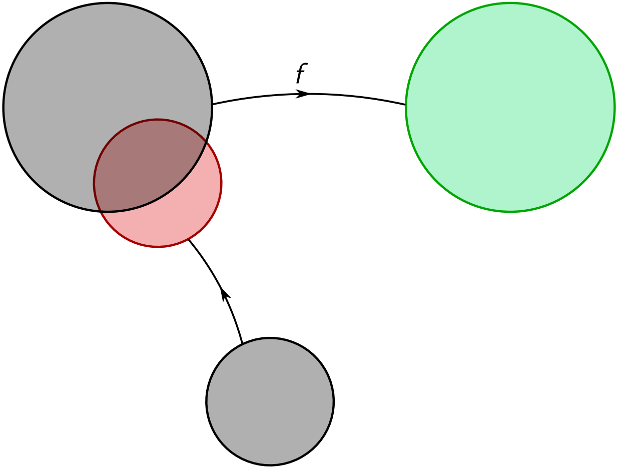

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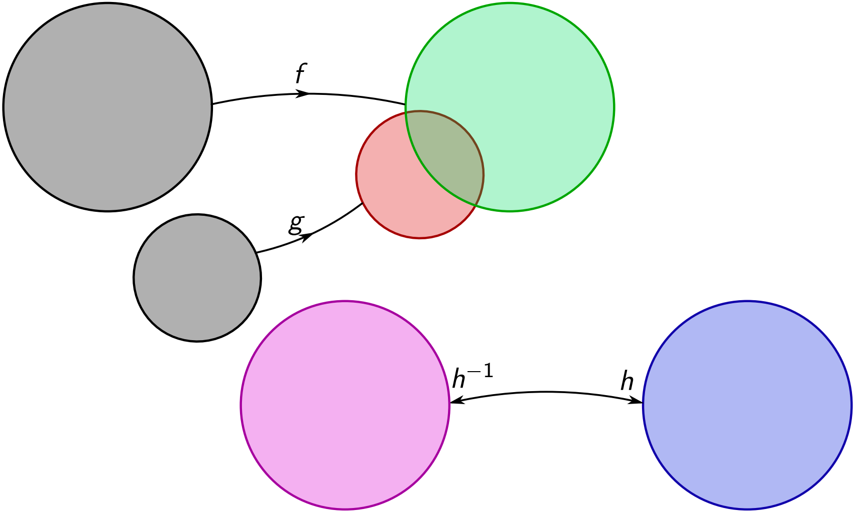

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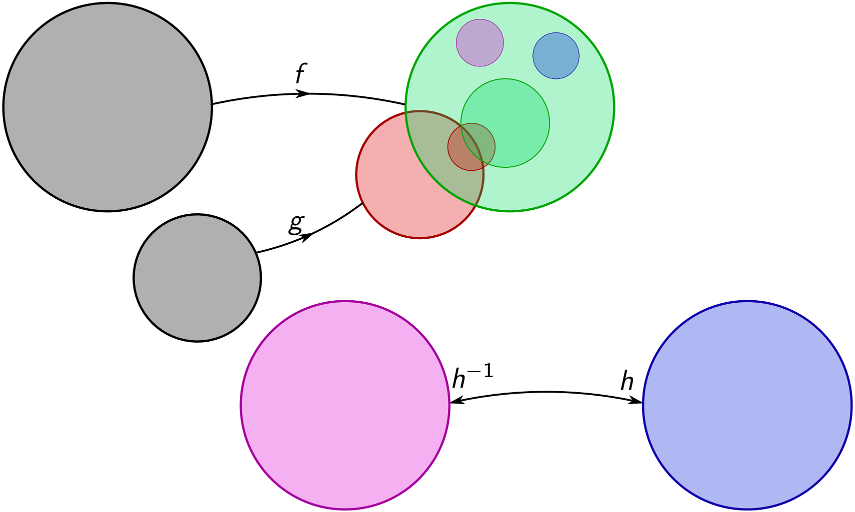

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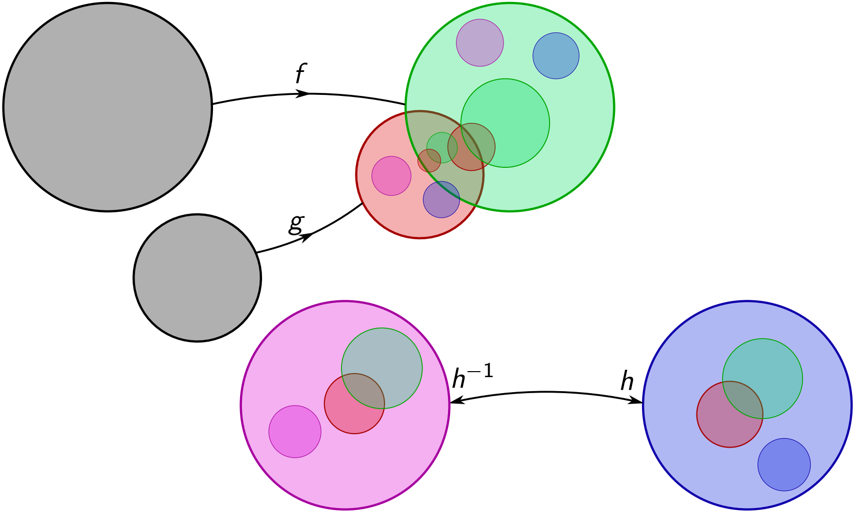

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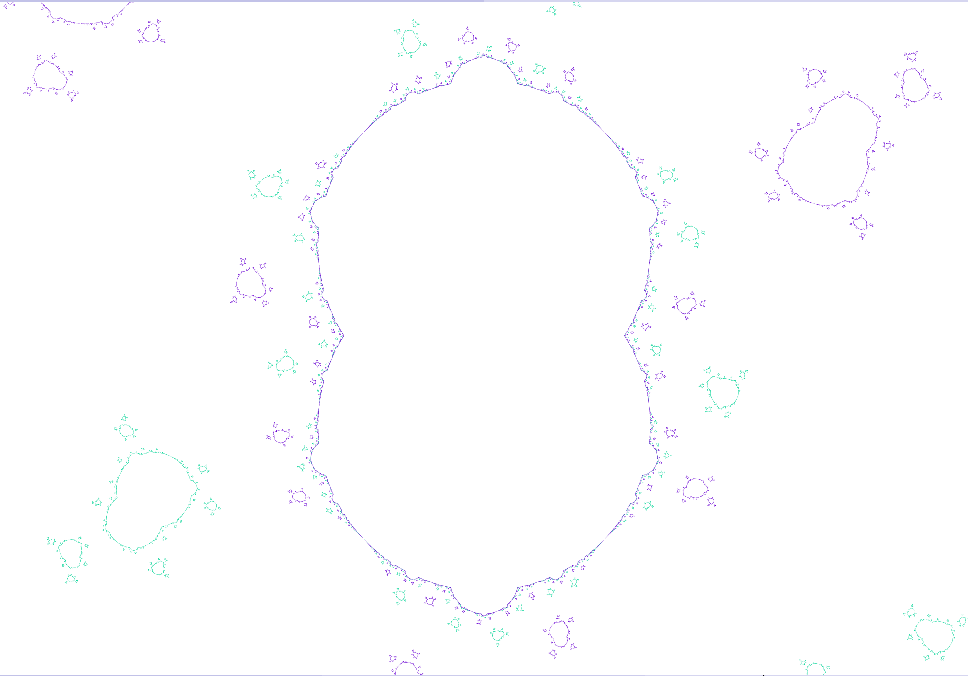

Composition sequences

Fix a set of Möbius transformations $\mathcal{F}$.

A *composition sequence* of Möbius transformations generated by $\mathcal{F}$ is any sequence with n$^{\text{th}}$ term

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n,$$

where each f_i is chosen from $\mathcal{F}$.

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Write $f(z) = \dfrac{1}{z+2}$ and $g(z) = \dfrac{3}{z+1}$. Then

$$F_1(z) = f(z) = \frac{1}{z+2}$$

$$F_2(z) = f \circ g(z) = \cfrac{1}{2 + \cfrac{3}{1+z}}$$

$$F_3(z) = f \circ g \circ g(z) = \cfrac{1}{2 + \cfrac{3}{1 + \cfrac{3}{1+z}}}$$

so that $F_n(0)$ is the n$^{\text{th}}$ convergent of some continued fraction.

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Escaping sequences

Definition

We say a sequence of Möbius transformations g_n is *escaping* if $g_n\zeta$ accumulates only on the boundary of hyperbolic space.
Equivalently

$$\rho(g_n\zeta, \zeta) \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty.$$

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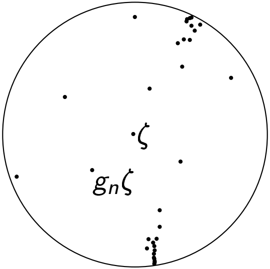

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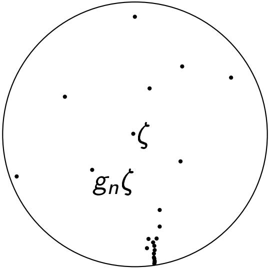

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Given a particular composition sequence, does it converge?

Related question:

Given a set of Möbius transformations $\mathcal{F}$ when does every composition sequence generated by $\mathcal{F}$

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Let $S = \langle \mathcal{F} \rangle$ be the semigroup generated by $\mathcal{F}$.

Proposition

Every composition sequence generated by $\mathcal{F}$ escapes if and only if $\mathrm{Id} \notin \overline{S}$.

Proposition

If Λ^+ and Λ^- are disjoint then every escaping composition sequence generated by $\mathcal{F}$ converges.

On the other hand:

Proposition

There is a dense G_δ set (w.r.t. the topology on Λ^-), D^- contained in Λ^- such that if Λ^+ meets D^-, then not every composition sequence generated by $\mathcal{F}$ converges.

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Theorem

Suppose $\mathcal{F}$ is bounded set of Möbius transformations acting on $\mathbb{B}^2$, generating a non-elementary semigroup S.

Every composition sequence drawn from $\mathcal{F}$ converges if and only if $\mathrm{Id} \notin \overline{S}$ and Λ^+ is not the whole of $\mathbb{S}^1$.

Lemma

If S is a semigroup of Möbius transformations acting on $\mathbb{B}^3$ such that $|\Lambda^-| > 1$ and if $\Lambda^- \subseteq \Lambda^+$, then there exists a composition sequence in S that does not converge.

Whenever $\Lambda^+ = \mathbb{S}^1$ there exists some composition sequence that does not converge.

Lemma

If S is a semigroup of Möbius transformations acting on $\mathbb{B}^3$ such that $|\Lambda^-| > 1$ and if $\Lambda^- \subseteq \Lambda^+$, then there exists a composition sequence in S that does not converge.

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Optimal?

Can we drop the reference to $\Lambda^+ = \mathbb{S}^1$, in other words is the following true?

Conjecture

Suppose $\mathcal{F}$ is bounded set of Möbius transformations acting on $\mathbb{B}^2$, generating a non-elementary semigroup S.
Every composition sequence drawn from $\mathcal{F}$ converges if and only if every composition sequence escapes.

Literature

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Thank you for your attention!