Fast escape using normal families

Dan Nicks

University of Nottingham

March 2015

Introduction

- Quasiregular functions on $\mathbb{R}^d$ generalize analytic functions on $\mathbb{C}$.
- We aim to prove iterative results for quasiregular maps analogous to complex dynamics.
- Today we'll first introduce quasiregular maps and then do some dynamics.

Quasiregular mappings

- Informally, a continuous sense-preserving map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is called quasiregular (qr) if it maps infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity (i.e. the ratio major axis/minor axis is bounded).
- Quasiregular $=$ quasiconformal without injectivity condition.
- For $K \geq 1$, we say that f is K-quasiregular if the amount of local stretching is $\leq K$ everywhere.

Quasiregular mappings

- Informally, a continuous sense-preserving map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is called quasiregular (qr) if it maps infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity (i.e. the ratio major axis/minor axis is bounded).

- Quasiregular $=$ quasiconformal without injectivity condition.

- For $K \geq 1$, we say that f is K-quasiregular if the amount of local stretching is $\leq K$ everywhere.

- Examples: $(x, y) \mapsto (Kx, y)$ is K-qr,
  for $K \in \mathbb{N}$, the map $re^{i\theta} \mapsto re^{iK\theta}$ is K-qr,
  analytic functions on $\mathbb{C}$ are 1-qr.

- A composition of qr maps is itself qr.

Properties of quasiregular maps

Why are qr maps on $\mathbb{R}^d$ a good generalization of entire functions on $\mathbb{C}$?

Theorem (Reshetnyak, 1967-68)

Non-constant quasiregular maps are discrete and open.

Properties of quasiregular maps

Why are qr maps on $\mathbb{R}^d$ a good generalization of entire functions on $\mathbb{C}$?

Theorem (Reshetnyak, 1967-68)

Non-constant quasiregular maps are discrete and open.

Rickman proved a Picard theorem for quasiregular maps:

Theorem (Rickman, 1980)

For $d \geq 2$ and $K \geq 1$ there exists a constant $C = C(d, K)$ with the following property:
every K-qr map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ that omits C values must be constant.

Properties of quasiregular maps

Why are qr maps on $\mathbb{R}^d$ a good generalization of entire functions on $\mathbb{C}$?

Theorem (Reshetnyak, 1967-68)

Non-constant quasiregular maps are discrete and open.

Rickman proved a Picard theorem for quasiregular maps:

Theorem (Rickman, 1980)

For $d \geq 2$ and $K \geq 1$ there exists a constant $C = C(d, K)$ with the following property:
every K-qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ that omits C values must be constant.

This value $C = C(d, K)$ is called *Rickman's constant*.
(Picard's theorem says $C(2, 1) = 2$.)

Montel's theorem on normal families

Let σ denote the spherical metric on $\overline{\mathbb{C}}$ or $\overline{\mathbb{R}^d}$. Recall:

Theorem (Montel)

Let $\mathcal{F}$ be a family of analytic functions on a domain $D \subset \mathbb{C}$.
Suppose $\exists \varepsilon > 0$ such that

(i) *each $f \in \mathcal{F}$ omits 2 values $a_1(f), a_2(f) \in \mathbb{C}$,*

(ii) *$\sigma(a_1(f), a_2(f)) \geq \varepsilon$ and $\sigma(a_i(f), \infty) \geq \varepsilon$.*

Then $\mathcal{F}$ is a normal family.

Miniowitz used Rickman's theorem to prove a quasiregular version...

Theorem (Miniowitz, 1982)

Let $\mathcal{F}$ be a family of K-qr maps on a domain $D \subset \mathbb{R}^d$ and let $C = C(d, K)$ be Rickman's constant. Suppose $\exists \varepsilon > 0$ such that

(i) *each $f \in \mathcal{F}$ omits C values $a_1(f), \ldots a_C(f) \in \mathbb{R}^d$,*

(ii) $\sigma(a_i(f), a_j(f)) \geq \varepsilon$ *and* $\sigma(a_i(f), \infty) \geq \varepsilon$.

Then $\mathcal{F}$ is a normal family.

Theorem (Miniowitz, 1982)

Let $\mathcal{F}$ be a family of K-qr maps on a domain $D \subset \mathbb{R}^d$ and let $C = C(d, K)$ be Rickman's constant. Suppose $\exists \varepsilon > 0$ such that

(i) *each $f \in \mathcal{F}$ omits C values $a_1(f), \ldots a_C(f) \in \mathbb{R}^d$,*

(ii) *$\sigma(a_i(f), a_j(f)) \geq \varepsilon$ and $\sigma(a_i(f), \infty) \geq \varepsilon$.*

Then $\mathcal{F}$ is a normal family.

- Note this requires a family of maps all K-qr with the same K.
- In general, if f is K-qr, then the iterate f^n is only K^n-qr. So Miniowitz–Montel cannot be applied to the family $\{f^n\}$!

Theorem (Miniowitz, 1982)

Let $\mathcal{F}$ be a family of K-qr maps on a domain $D \subset \mathbb{R}^d$ and let $C = C(d, K)$ be Rickman's constant. Suppose $\exists \varepsilon > 0$ such that

(i) *each $f \in \mathcal{F}$ omits C values $a_1(f), \dots a_C(f) \in \mathbb{R}^d$,*

(ii) *$\sigma(a_i(f), a_j(f)) \geq \varepsilon$ and $\sigma(a_i(f), \infty) \geq \varepsilon$.*

Then $\mathcal{F}$ is a normal family.

- Note this requires a family of maps all K-qr with the same K.
- In general, if f is K-qr, then the iterate f^n is only K^n-qr. So Miniowitz–Montel cannot be applied to the family $\{f^n\}$!
- We can still make use of Miniowitz–Montel by applying it instead to a family of rescalings. For example, if f is K-qr, then

$$\left\{ x \mapsto \frac{f(rx)}{s} : r > 0,\ s > 0 \right\} \text{ is a family of } K\text{-qr maps.}$$

Polynomial type vs transcendental type

Definition

A qr map f is said to be of *polynomial type* if $\lim_{x\to\infty} |f(x)| = \infty$.
Otherwise, this limit does not exist and f is *transcendental type*.

Polynomial type vs transcendental type

Definition

A qr map f is said to be of *polynomial type* if $\lim_{x\to\infty} |f(x)| = \infty$. Otherwise, this limit does not exist and f is *transcendental type*.

Denote the *maximum modulus* by

$$M(r, f) = M(r) := \max_{|x|=r} |f(x)|.$$

Lemma (Bergweiler, 2006)

Let f be quasiregular of transcendental type and let $A > 1$. Then

$$\lim_{r\to\infty} \frac{M(Ar, f)}{M(r, f)} = \infty.$$

Dynamics — Escape from $\mathbb{C}$

For a function $f\colon \mathbb{R}^d \to \mathbb{R}^d$, we consider the *escaping set*

$$I(f) = \{x \in \mathbb{R}^d : f^n(x) \to \infty\}.$$

Theorem (Eremenko, 1989)

If $f\colon \mathbb{C} \to \mathbb{C}$ is transcendental entire, then $I(f) \neq \emptyset$.

Dynamics — Escape from $\mathbb{C}$

For a function $f\colon \mathbb{R}^d \to \mathbb{R}^d$, we consider the *escaping set*

$$I(f) = \{x \in \mathbb{R}^d : f^n(x) \to \infty\}.$$

Theorem (Eremenko, 1989)

If $f\colon \mathbb{C} \to \mathbb{C}$ is transcendental entire, then $I(f) \neq \emptyset$.

How fast can orbits escape to ∞?

- The iterated max modulus provides an upper bound:

$$\text{if } |x| \leq R, \text{ then } |f^n(x)| \leq M^n(R).$$

- But points $|x| > R$ can go faster

Dynamics — Escape from $\mathbb{C}$

For a function $f\colon \mathbb{R}^d \to \mathbb{R}^d$, we consider the *escaping set*

$$I(f) = \{x \in \mathbb{R}^d : f^n(x) \to \infty\}.$$

Theorem (Eremenko, 1989)

If $f\colon \mathbb{C} \to \mathbb{C}$ is transcendental entire, then $I(f) \neq \emptyset$.

How fast can orbits escape to ∞?

- The iterated max modulus provides an upper bound:

$$\text{if } |x| \leq R, \text{ then } |f^n(x)| \leq M^n(R).$$

- But points $|x| > R$ can go faster — Eremenko's method gives:

Theorem (Bergweiler, Hinkkanen, 1999)

Let f be trans entire and $R > 0$. Then there exists $z \in \mathbb{C}$ such that

$$|f^n(z)| \geq M^n(R) \quad \textit{for all } n.$$

Fast escape for quasiregular maps

Theorem (Bergweiler, Fletcher, Drasin, 2014)

Let f be trans type qr and $R > 0$. Then there exists $x \in \mathbb{R}^d$ such that

$$|f^n(x)| \geq M^n(R) \quad \text{for all } n.$$

This theorem follows from the next covering lemma by pulling back.
We use the notation $A(r, s) = \{x \in \mathbb{R}^d : r < |x| < s\}$.

Fast escape for quasiregular maps

Theorem (Bergweiler, Fletcher, Drasin, 2014)

Let f be trans type qr and $R > 0$. Then there exists $x \in \mathbb{R}^d$ such that

$$|f^n(x)| \geq M^n(R) \quad \textit{for all } n.$$

This theorem follows from the next covering lemma by pulling back. We use the notation $A(r, s) = \{x \in \mathbb{R}^d : r < |x| < s\}$.

Lemma (BFD)

Let f be trans type qr. For all large r, there exists $S \geq M(r, f)$ such that

$$f(A(r, 4r)) \supset A(S, 4S).$$

We'll see how to prove this lemma using normal families.

Let f be trans type qr and let C be Rickman's constant.

Claim: For all large r, the image $f(A(r, 4r))$ contains

$$A(4^{2p}M(2r), 4^{2p+1}M(2r)) \quad \text{for some } p \in \{0, 1, \dots, C-1\}.$$

Let f be trans type qr and let C be Rickman's constant.

Claim: For all large r, the image $f(A(r,4r))$ contains

$$A(4^{2p}M(2r), 4^{2p+1}M(2r)) \quad \text{for some } p \in \{0, 1, \ldots, C-1\}.$$

To prove this claim, we consider the family of rescalings

$$\left\{ g_r(x) = \frac{f(rx)}{M(2r, f)} : r > 0 \right\}.$$

Let f be trans type qr and let C be Rickman's constant.

Claim: For all large r, the image $f(A(r, 4r))$ contains

$$A(4^{2p}M(2r), 4^{2p+1}M(2r)) \quad \text{for some } p \in \{0, 1, \ldots, C-1\}.$$

To prove this claim, we consider the family of rescalings

$$\left\{ g_r(x) = \frac{f(rx)}{M(2r, f)} : r > 0 \right\}.$$

- If the claim fails, $\exists r_k \to \infty$ such that on $A(1, 4)$ each rescaled function g_{r_k} omits at least one point in each of the C annuli

$$A(1, 4),\ A(4^2, 4^3),\ \ldots,\ A(4^{2(C-1)}, 4^{2(C-1)+1}).$$

- Then Miniowitz–Montel $\Longrightarrow \{g_{r_k}\}$ is a normal family on $A(1, 4)$.

$$g_r(x) = \frac{f(rx)}{M(2r, f)}$$

On the other hand, the family $\{g_{r_k}\}$ cannot be normal on $A(1,4)$ because:

$$M(3, g_{r_k}) = \frac{M(3r_k, f)}{M(2r_k, f)} \to \infty \quad \text{as } r_k \to \infty;$$

but also,

$$\text{for } |x| \leq 2, \text{ we have } |g_{r_k}(x)| = \frac{|f(r_k x)|}{M(2r_k, f)} \leq 1.$$

This proves the claim (and so the qr fast escape theorem).

Slow escape

Slow escape

There are always points that escape arbitrarily slowly. The following was first proved for trans entire functions on $\mathbb{C}$ by Rippon and Stallard.

Theorem (N.)

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular of trans type.
Take a positive sequence $a_n \to \infty$ (however slowly).
Then there exists $x \in I(f)$ such that $|f^n(x)| \le a_n$ for all large n.

Slow escape

There are always points that escape arbitrarily slowly. The following was first proved for trans entire functions on $\mathbb{C}$ by Rippon and Stallard.

Theorem (N.)

Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular of trans type.
Take a positive sequence $a_n \to \infty$ (however slowly).
Then there exists $x \in I(f)$ such that $|f^n(x)| \leq a_n$ for all large n.

The entire and qr cases use covering lemmas in similar ways, but the proofs are quite different.

An unexpected corollary of part of the new qr proof is the next result, which is new even for entire fns and is (kind of) 'fast' ...

Question For a general f, can we find points with modulus $\approx R$ that escape to infinity at a rate $\approx M^n(R, f)$?

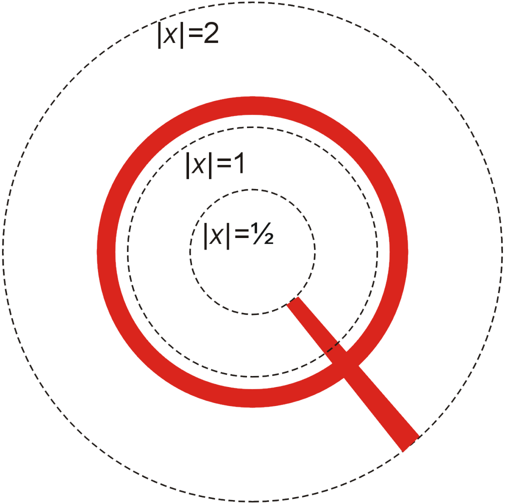
Question For a general f, can we find points with modulus $\approx R$ that escape to infinity at a rate $\approx M^n(R, f)$?

Theorem (N.)

Let f be quasiregular of trans type.
For all large $R > 0$, there exists $x \in \mathbb{R}^d$ with $\frac{R}{2} \leq |x| \leq 2R$ and

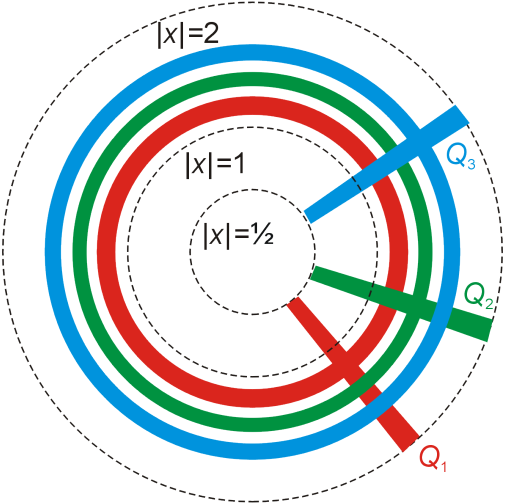
$$\lim_{n\to\infty} \frac{|f^n(x)|}{M^n(R)} = 1.$$

The key to the proof is a covering result where the image of a set "of scale r" covers one "of scale $M(r)$".
To do this, we next introduce some sets with a particular shape.



Fix K, d and let $C = C(d, K)$.

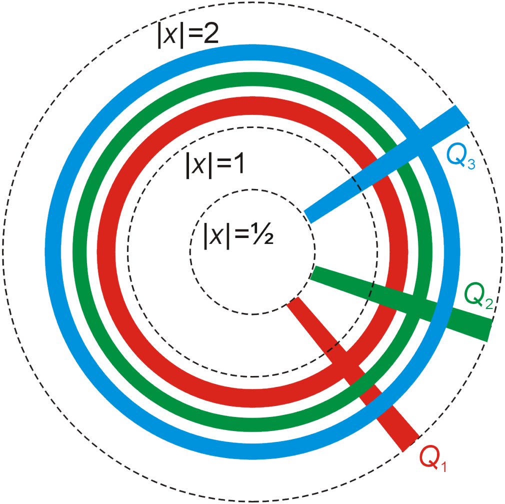
Introduce sets ...



Fix K, d and let $C = C(d, K)$.

Introduce sets $Q_1, \ldots, Q_{2C}$.

Nowhere do 3 of these overlap.



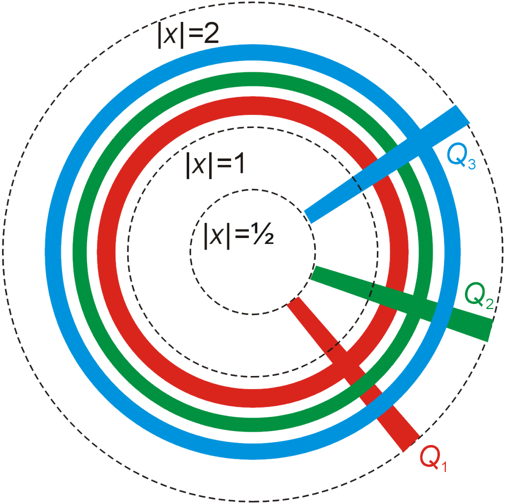
Fix K, d and let $C = C(d, K)$.

Introduce sets $Q_1, \ldots, Q_{2C}$.

Nowhere do 3 of these overlap.

For $r > 0$, let

$$Q_j(r) := rQ_j.$$



Fix K, d and let $C = C(d, K)$.

Introduce sets $Q_1, \ldots, Q_{2C}$.

Nowhere do 3 of these overlap.

For $r > 0$, let

$$Q_j(r) := rQ_j.$$

Q-lemma (N.)

If f is K-quasiregular on $\mathbb{R}^d$ of trans type, then for all large r and each j,

$$f(Q_j(r)) \supset Q_p(M(r)) \quad \text{for some } p \in \{1, \ldots, 2C\}.$$

Q-lemma $\implies$ controlled rate of escape

Repeatedly using the Q-lemma gives a sequence of Q-sets where the image of each one under f covers the next:

$$Q_1(R) \overset{\text{image covers}}{\longrightarrow} Q_{j_1}(M(R)) \overset{\text{image covers}}{\longrightarrow} Q_{j_2}(M^2(R)) \longrightarrow Q_{j_3}(M^3(R)) \longrightarrow \ldots$$

Now a pullback argument gives a point $x \in Q_1(R)$ such that $f^n(x) \in Q_{j_n}(M^n(R))$ for all n, i.e.

$$\frac{R}{2} \leq |x| \leq 2R \quad \text{and} \quad \frac{1}{2} \leq \frac{|f^n(x)|}{M^n(R)} \leq 2.$$

Q-lemma $\Longrightarrow$ controlled rate of escape

Repeatedly using the Q-lemma gives a sequence of Q-sets where the image of each one under f covers the next:

$$Q_1(R) \stackrel{\text{image covers}}{\longrightarrow} Q_{j_1}(M(R)) \stackrel{\text{image covers}}{\longrightarrow} Q_{j_2}(M^2(R)) \longrightarrow Q_{j_3}(M^3(R)) \longrightarrow \ldots$$

Now a pullback argument gives a point $x \in Q_1(R)$ such that $f^n(x) \in Q_{j_n}(M^n(R))$ for all n, i.e.

$$\frac{R}{2} \leq |x| \leq 2R \quad \text{and} \quad \frac{1}{2} \leq \frac{|f^n(x)|}{M^n(R)} \leq 2.$$

To get $|f^n(x)| \sim M^n(R)$ we need to work a little harder (squeeze the Q-sets into thinner annuli).

Q-lemma (N.)

If f is K-quasiregular on $\mathbb{R}^d$ of trans type, then for all large r and each j,

$$f(Q_j(r)) \supset Q_p(M(r)) \quad \text{for some } p \in \{1, \ldots, 2C\}.$$

To prove the Q-lemma, consider the rescaled family

$$\left\{ h_r(x) = \frac{f(rx)}{M(r, f)} : r > 0 \right\}.$$

If the Q-lemma fails, then for some j and some $r_k \to \infty$

- the image $f(Q_j(r_k))$ omits a point in $Q_p(M(r_k))$, for every p,

Q-lemma (N.)

If f is K-quasiregular on $\mathbb{R}^d$ of trans type, then for all large r and each j,

$$f(Q_j(r)) \supset Q_p(M(r)) \quad \text{for some } p \in \{1, \ldots, 2C\}.$$

To prove the Q-lemma, consider the rescaled family

$$\left\{ h_r(x) = \frac{f(rx)}{M(r,f)} : r > 0 \right\}.$$

If the Q-lemma fails, then for some j and some $r_k \to \infty$

- the image $f(Q_j(r_k))$ omits a point in $Q_p(M(r_k))$, for every p,
- rescaling, $h_{r_k}(Q_j)$ omits a point in each of $Q_1, \ldots, Q_{2C}$,
- then Miniowitz–Montel $\Longrightarrow \{h_{r_k}\}$ is normal on Q_j.

Q-lemma (N.)

If f is K-quasiregular on $\mathbb{R}^d$ of trans type, then for all large r and each j,

$$f(Q_j(r)) \supset Q_p(M(r)) \quad \text{for some } p \in \{1, \ldots, 2C\}.$$

To prove the Q-lemma, consider the rescaled family

$$\left\{ h_r(x) = \frac{f(rx)}{M(r, f)} : r > 0 \right\}.$$

If the Q-lemma fails, then for some j and some $r_k \to \infty$

- the image $f(Q_j(r_k))$ omits a point in $Q_p(M(r_k))$, for every p,
- rescaling, $h_{r_k}(Q_j)$ omits a point in each of $Q_1, \ldots, Q_{2C}$,
- then Miniowitz–Montel $\implies \{h_{r_k}\}$ is normal on Q_j.

(Note that, although the Q-sets overlap, we've taken $2C$ of them to be sure of finding C well-spaced omitted values.)

But $\{h_{r_k}\}$ cannot be a normal family on Q_j because

- for $|x| \leq 1$, we have $|h_r(x)| = \dfrac{|f(rx)|}{M(r,f)} \leq 1$,
- Q_j contains a max mod point x with $|x| = A > 1$ at which

$$|h_r(x)| = \frac{M(Ar,f)}{M(r,f)} \to \infty \quad \text{as } r \to \infty.$$

This proves the Q-lemma.