Connectedness properties of the set where the iterates of an entire function are unbounded

John Osborne

(joint work with Phil Rippon and Gwyneth Stallard)

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The set of points whose orbits are unbounded

- f an entire function, $f^n = f \circ f \circ \ldots \circ f$
- $(f^n(z))_{n \in \mathbb{N}}$ the *orbit* of z under f
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- $K(f)^c$ is connected

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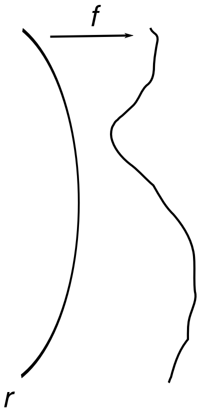
- $K(f)^c \setminus I(f) \neq \emptyset$
- When is $K(f)^c$ connected?

Iterating the minimum modulus function

Define

- $m(r) = m(r, f) := \min\{|f(z)| : |z| = r\}$
- $m^n(r)$ to be the nth iterate of the function $r \mapsto m(r)$.



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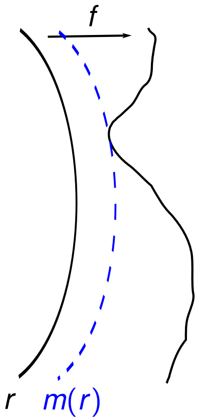

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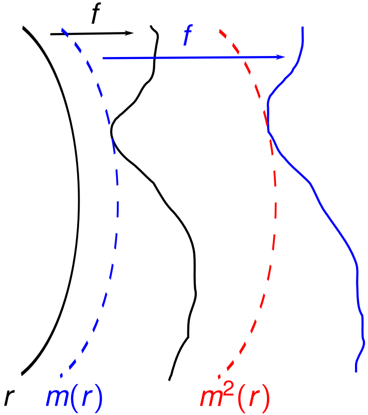

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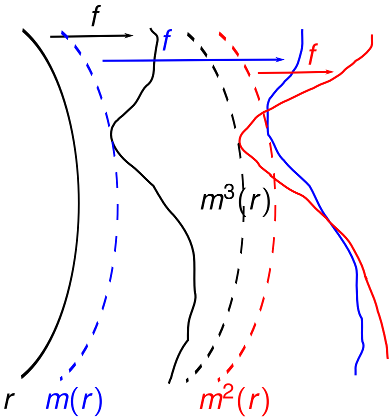

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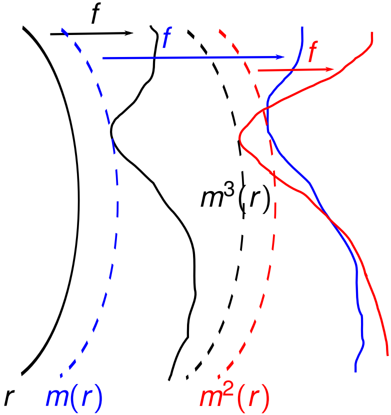

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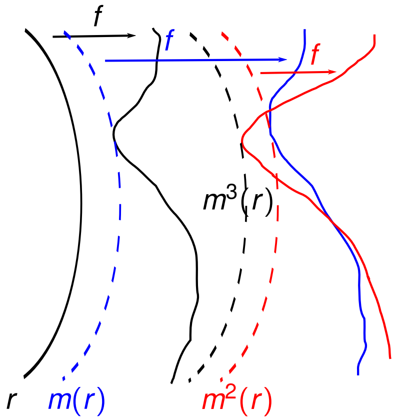

For a transcendental entire function (compare the iteration of $M(r)$):

- $\nexists\, R > 0$ with $m(r) > r \; \forall\, r \geq R$
- we can't always find $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$.

Some functions for which $K(f)^c$ is connected

Theorem A

Let f be a transcendental entire function for which there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$. Then $K(f)^c$ is connected.

Some functions for which $K(f)^c$ is connected

Theorem A

Let f be a transcendental entire function for which there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$. Then $K(f)^c$ is connected.

Theorem B

Let f be a transcendental entire function of order less than $\frac{1}{2}$. Then there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$, and therefore $K(f)^c$ is connected.

Recall that the *order* ρ of a transcendental entire function is defined as

$$\rho := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

An idea of the proof of Theorem A

Suppose $K(f)^c$ is disconnected.

Lemma

A subset X of $\mathbb{C}$ is disconnected if and only if there exists a closed, connected set $\Gamma \subset X^c$ such that at least two different components of Γ^c intersect X.

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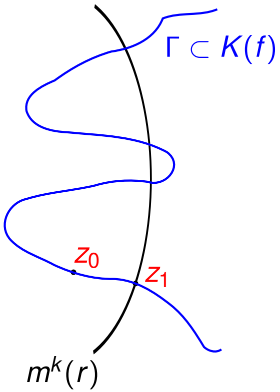
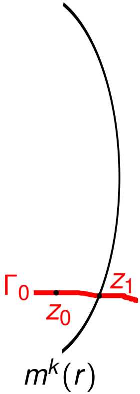


An idea of the proof of Theorem A

Suppose we have a continuum $\Gamma_0 \subset K(f)$ such that

- for some $z_0 \in \Gamma_0$, $|f^n(z_0)| < m^k(r)$ for all $n \in \mathbb{N}$, and
- $\exists\, z_1 \in \Gamma_0 \cap \{z : |z| = m^k(r)\}$.



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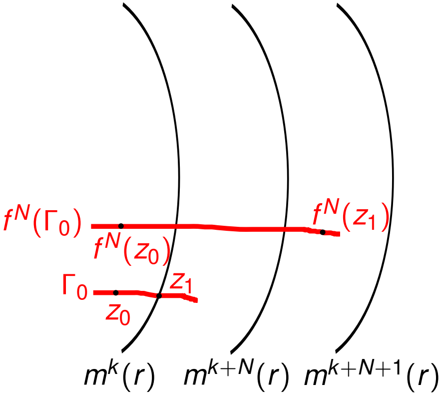

$N \geq 1$ is the largest integer such that $|f^N(z_1)| \geq m^{k+N}(r)$.

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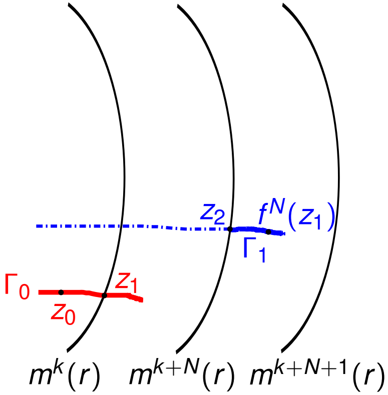

Choose $\Gamma_1 \subset f^N(\Gamma_0)$ so that it contains a point z_2 with modulus $m^{k+N}(r)$ but no points of smaller modulus.

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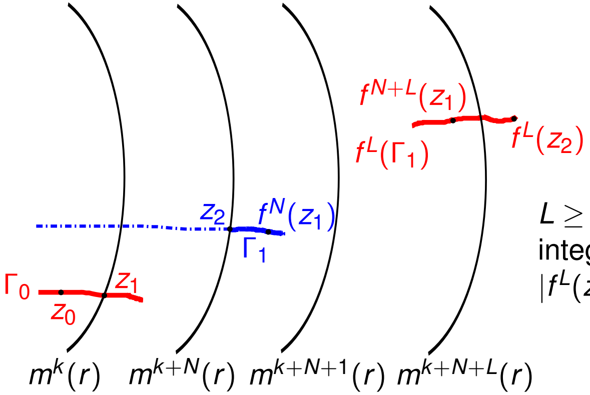

$L \geq 1$ is the largest integer such that $|f^L(z_2)| \geq m^{k+N+L}(r)$.

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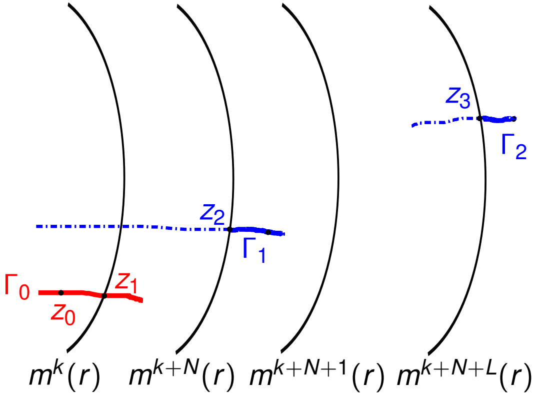
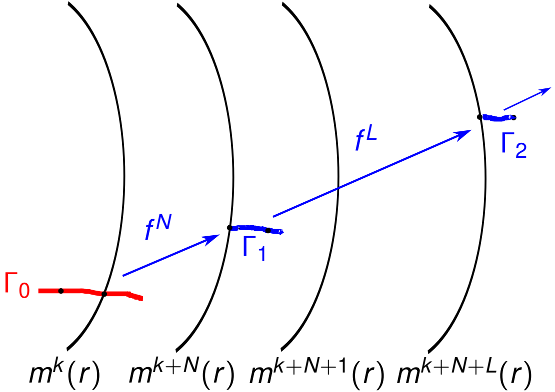


Choose $\Gamma_2 \subset f^L(\Gamma_1)$ so that it contains a point z_3 with modulus $m^{k+N+L}(r)$ but no points of smaller modulus.

An idea of the proof of Theorem A

We have constructed a sequence (Γ_n) of compact sets such that $f^{k_n}(\Gamma_n) \supset \Gamma_{n+1}$ for some (k_n).



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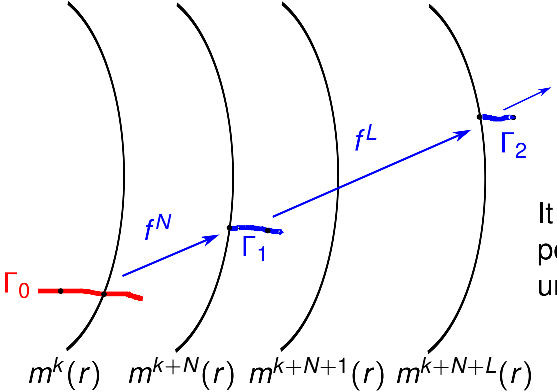
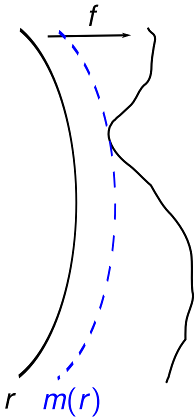


It follows that there is a point in Γ_0 with unbounded orbit. #

Generalising the condition in Theorem A

'... there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$.'



We have:

- a sequence of nested discs $\{z : |z| < m^n(r)\}$
- that fill the plane
- such that each boundary circle is mapped outside the next disc in the sequence.

Can we replace the discs by arbitrary bounded, simply connected domains?

A more general result

Theorem C

Let f be a transcendental entire function, and $(D_n)_{n\in\mathbb{N}}$ be a sequence of bounded, simply connected domains such that

(a) $f(\partial D_n)$ *surrounds* D_{n+1}, *for* $n \in \mathbb{N}$, *and*

(b) *every disc centred at* 0 *is contained in* D_n *for sufficiently large* n.

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Is this really more general than Theorem A?

Example: Let $f(z) = -10ze^{-z} - \frac{1}{2}z$.
Note that $m(r) \sim \frac{1}{2}r$ as $r \to \infty$, so Theorem A does not hold.

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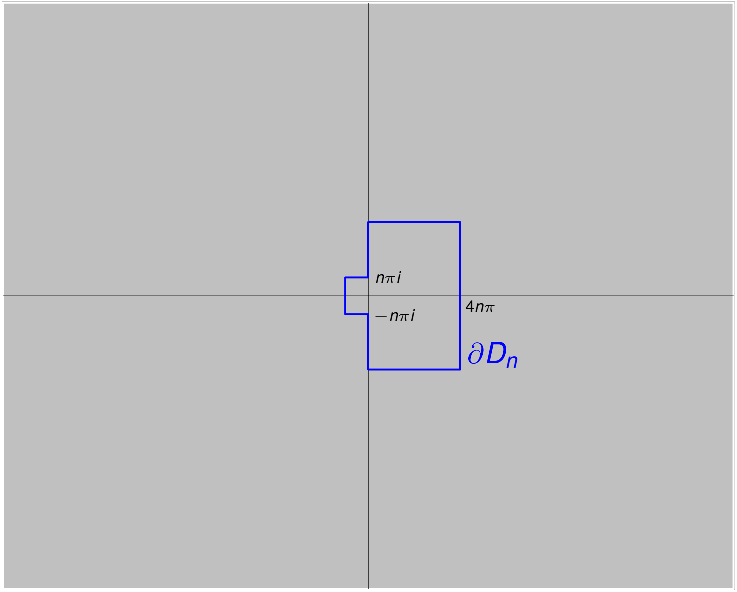

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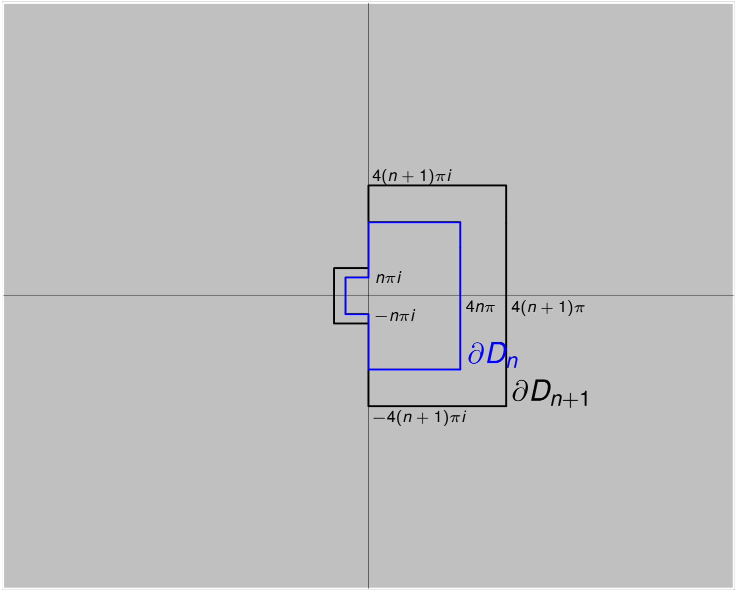

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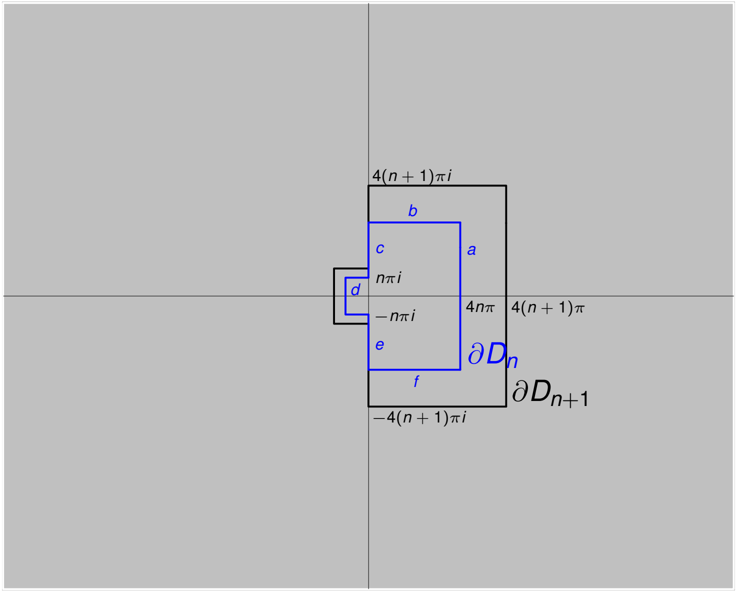

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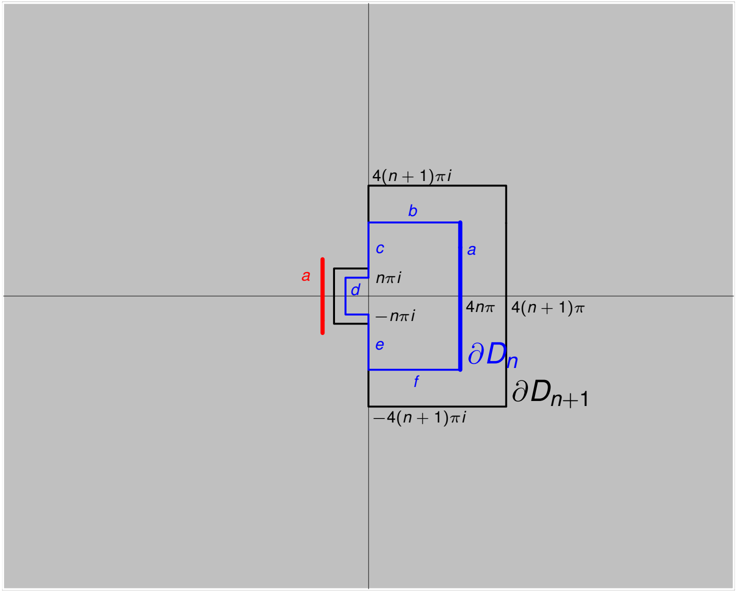

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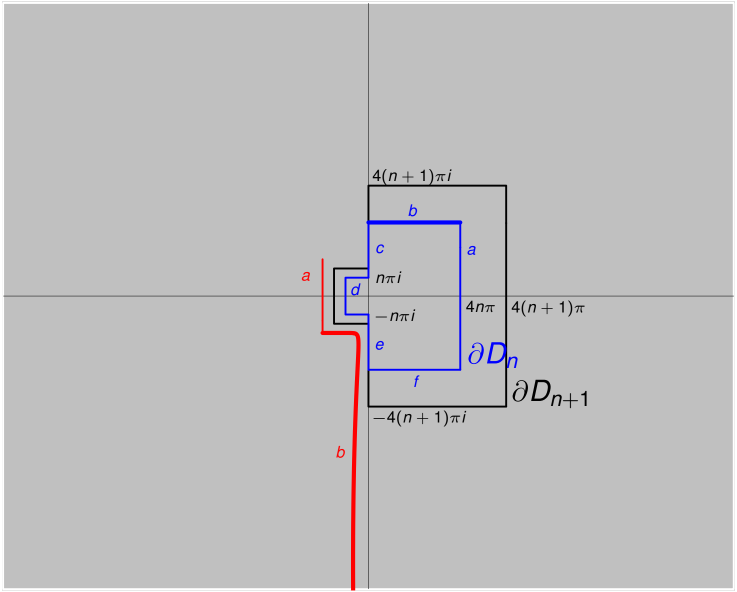

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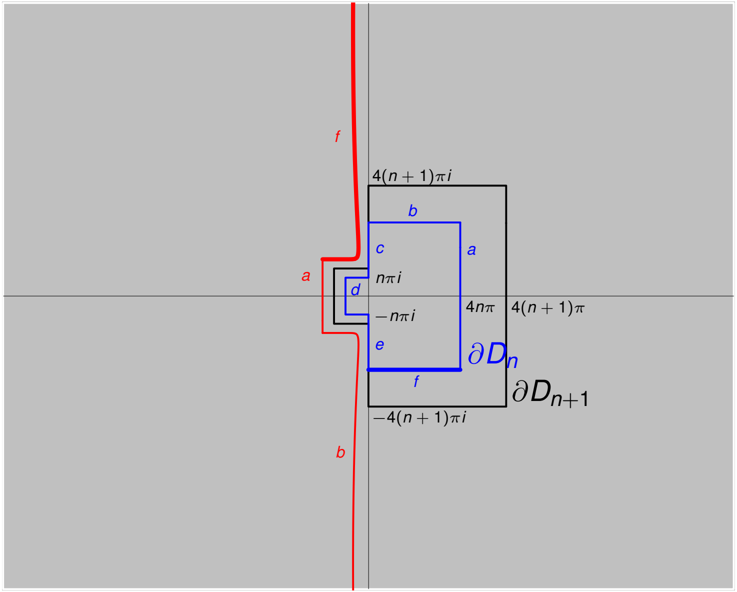

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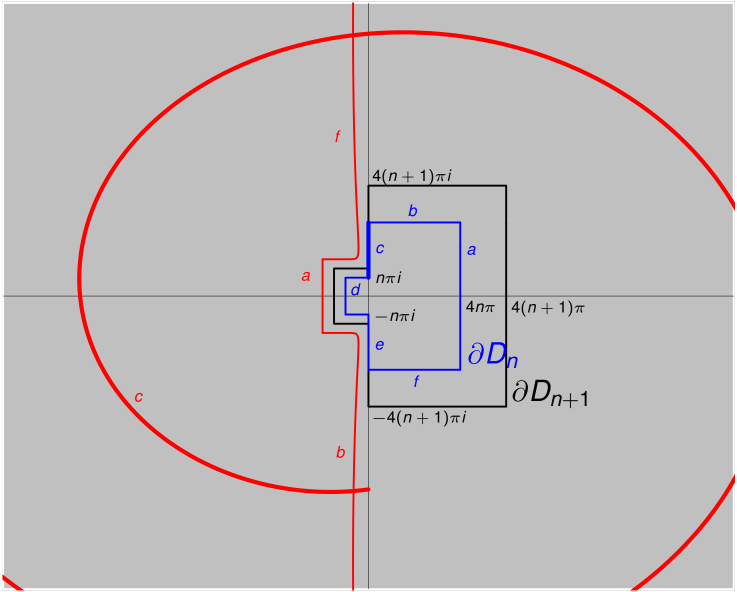

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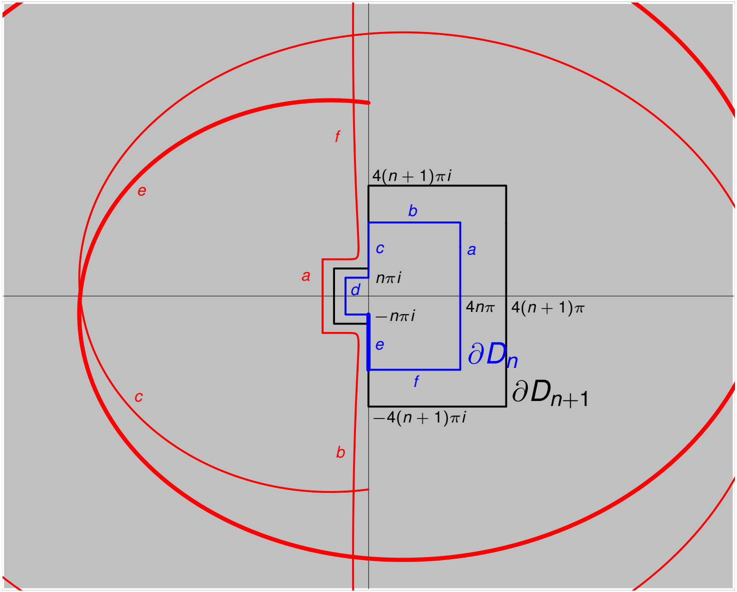

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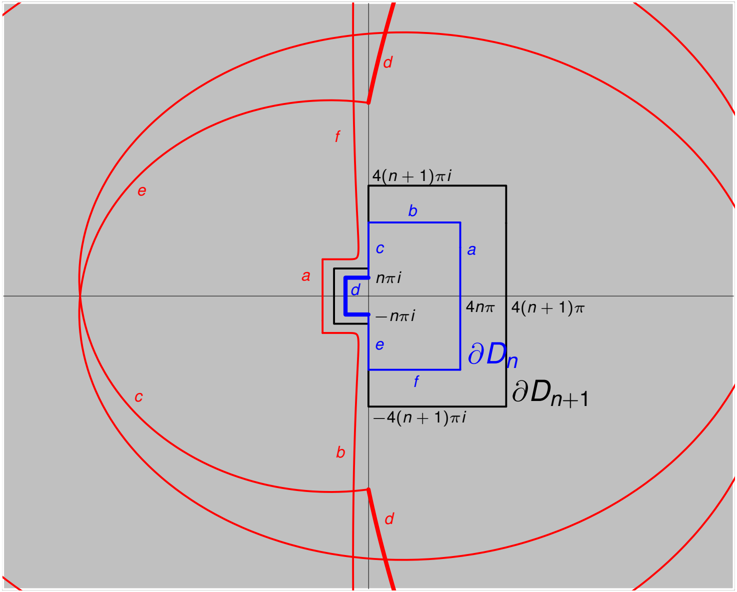

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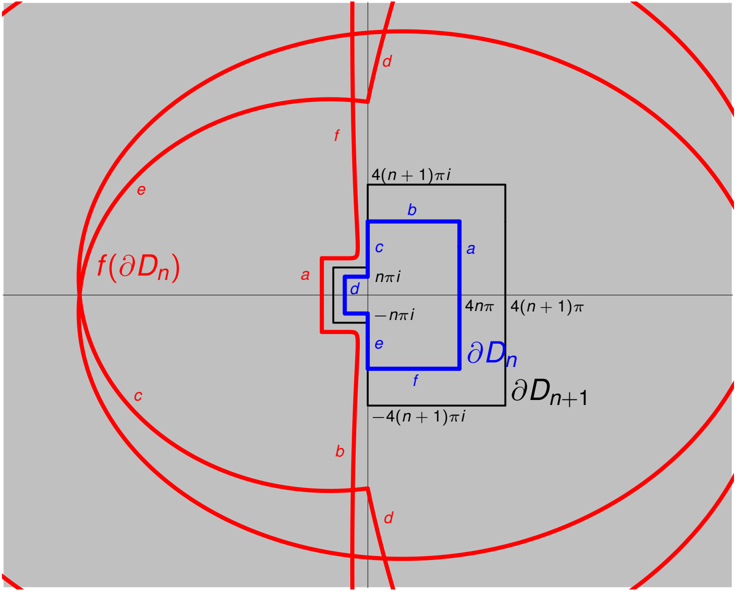

Further connectedness properties of $K(f)^c$

Theorem D

Let f be a transcendental entire function. Then:

(a) $K(f)^c \cup \{\infty\}$ *is connected.*

(b) *Either $K(f)^c$ is connected, or else every neighbourhood of a point in $J(f)$ meets uncountably many components of $K(f)^c$.*

(c) *If $I(f)$ is connected, then $K(f)^c$ is connected.*

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