Persistent Markov Partitions in Complex Dynamics

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There is surprisingly little early literature on the simpler case of expanding dynamical systems.

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The intersection of any finite set of open dense sets is itself open and dense, and the union of any finite set of nowhere dense sets is nowhere dense. So from any set $\mathcal{P}$ which satisfies the first three conditions we can find a set of (possibly smaller) sets which satisfies all four conditions, simply by replacing $\mathcal{P}$ by

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A set satisfying the first two conditions is a covering of X by closed sets. If $\mathcal{P}$ is a covering by closed sets then we can make a set of sets $\mathcal{Q}$ satisfying condition 3 also by

$$\mathcal{Q} = \left\{ \begin{array}{l} P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_s} : \\ \mathrm{int}(P_{i_1}) \cap \cdots \cap \mathrm{Int}(P_{i_s}) \neq \emptyset, \\ s \text{ maximal} \end{array} \right\}$$

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If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of X then $\mathcal{P} \vee \mathcal{Q}$ is also a partition of X where

$$\mathcal{P} \vee \mathcal{Q} = \{\overline{\mathrm{int}(P) \cap \mathrm{int}(Q)} : P \in \mathcal{P},\ Q \in \mathcal{Q}\}.$$

The Markov property

We shall say that a partition or covering $\mathcal{P} = \{P_i : 1 \le i \le r\}$ of X, is Markov respect to a map $f : X \to X$ if whenever $\mathrm{int}(P_i) \cap f(P_j) \ne \emptyset$ then

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Given a Markov covering, the partition constructed from the process outlined above is Markov.

Expanding maps

The definition of Markov partition given is particularly useful for expanding maps of compact metric spaces. A map $f : (X, d) \to (X, d)$ is expanding if there is $\lambda > 1$ and $\delta > 0$ such that $d(f(x), f(y)) \geq \lambda d(x, y)$ whenever x, $y \in X$ with $d(x, y) \leq \delta$.

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An expanding map is locally injective. An expanding map of a compact space is boundedly finite-to-one.

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$$x \in \mathrm{int}(P_i) \cap f^{-1}(P_j) \wedge d(x, y) < \delta_0 \wedge y \in f^{-1}(P_j)$$

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If $f : X \to X$ is continuous , and the sets in $\mathcal{P}$ are connected, then $\mathcal{P}$ is Markov for f if, for all i and j, P_i contains every component of $f^{-1}(P_j)$ which intersects $\text{int}(P_i)$.

The alternative definitions of Markov form the basis of existence results. The following result is essentially folklore, but I have not been able to find an early reference. (as already remarked, the focus in the 1960's and '70's was on invertible systems.) A proof can be found in [P-U].

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Let (X, d) be compact metric and let $f : (X, d) \to (X, d)$ be expanding. Then there are Markov partitions for f of arbitrarily small diameter.

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As one would expect, the proof is by a limiting process. One starts with a partition $\mathcal{P}_0$ of small diameter. Inductively one makes a partition $\mathcal{P}_{n+1}$ by taking unions of suitable parts of the inverse images of the sets in $\mathcal{P}_n$. The expanding property of f leads to geometric convergence of the sequence of partitions $\mathcal{P}_n$, in the Hausdorff topology.

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But even in two dimensions, a Hausdorff limit of a sequence of closed topological discs might not be a closed topological disc.

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The process of finding a graph G with this invariance property, starting with an initial choice of graph G_0, involves constructing a sequence G_n of graphs with $G_{n+1} \subset f^{-1}(G_n)$. It is natural to arrange that the graphs G_n are all homeomorphic, but it is not so easy, in general, to ensure that G_n converges to a graph G which is homeomorphic to G_n (for all n).

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But this is true if there is a sequence of continuous injective maps $\varphi_n : G_0 \to G_n$ and it is possible to show that the sequence φ_n converges uniformly to an injective map, because a continuous injective map from a compact space to a Hausdorff space is a homeomorphism.

Theorem

(F.T. Farrell and L.E. Jones, TAMS 1979) Let $f : X \to X$ *be an expanding map on a compact Riemannian surface* X*. Let* G_0 *be a graph satisfying mild combinatorial conditions. Let* $\varepsilon > 0$ *be given. Then there exists an integer* N *and a graph* G *within distance* ε *of* G_0 *such that* $G \subset f^{-N}(G)$*.*

The proof, is, once again, by a limiting process, with a sequence of continuous injective maps $\varphi_n : G_0 \to G_n$ for $n \geq 1$, and satisfying $f^N \circ \varphi_{n+1} = \varphi_n \circ f^N \circ h$ for a homeomorphism

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Theorem

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map such that every critical point is in the Fatou set, and such that the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Let F_0 be the union of the periodic Fatou components, and let $G_0 \subset \overline{\mathbb{C}} \setminus \overline{F_0}$ be a connected piecewise C^1 graph satisfying certain (mild) combinatorial conditions. Let U be a neighbourhood of G_0

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Then there exist an integer N and a graph $G' \subset U$ which is isotopic to G_0 in U and such that $G' \subset f^{-N}(G')$.

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The set

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We then analyse the accumulation points of such sets.

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Then we choose a graph $\Gamma_0 \subset X_0$ and $\Gamma_1 \subset f^{-1}(\Gamma_0$ and a homeomorphism $k_1 : \Gamma_0 \to \Gamma_1$, where the arcs of Γ_0 depend on the boundaries of the sets in the Markov partition.

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Then we choose a graph $\Gamma_0 \subset X_0$ and $\Gamma_1 \subset f^{-1}(\Gamma_0$ and a homeomorphism $k_1 : \Gamma_0 \to \Gamma_1$, where the arcs of Γ_0 depend on the boundaries of the sets in the Markov partition.
So is the homeomorphism k_1, which maps $\Gamma_0 \cap P$ to $\Gamma_1 \cap P$, for each set P in the Markov partition.

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Proof of injectivity of $\psi = \lim_{n\to\infty} \psi_n$ is then similar to the proof of injectivity of $\lim_{n\to\infty} \varphi_n$ in the original theorem of Farrell and Jones.

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But hyperbolic maps are stable on their Julia sets. In the same vein, for the map f as in our theorem, for which parabolic cycles are allowed, are expanding in a neighbourhood of the graph constructed.

This means that there is a local topological conjugacy and the graph, varies isotopically on a neighbourhood of f.

It is well-known that an expanding map f are stable in the sense that maps that are C^1 to f are topologically conjugate to f. Rational maps are never globally stable (although in some cases they are so for a suitable choice of metric with singularities).

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A natural first question is: which neighbourhood?

Yoccoz Puzzle

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The basic Markov partition in the Yoccoz puzzle for a quadratic polynomial with connected Julia set, outside the main cardioid, is the partition whose boundaries are formed by dynamical rays landing at the α fixed point.

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The partition can be regarded as a partition of the entire parameter space, which identifies with the complex plane $\mathbb{C}$. The boundary of the partition then consists of the set of c such that the critical value c lies in a union of two dynamical rays of rational argument (the boundary of one of the sets in the dynamical partition) and a single parabolic parameter value c.

Lemma

Let f be a rational map. Let $G\overline{\mathbb{C}}$ be a graph such that $G \subset f^{-1}(G)$. Suppose that there is an open neighbourhood U of G which is disjoint from the critical values of f and such that U contains the closure of any components of $f^{-1}(U)$ which intersect G.

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Then for g near f, there is a graph $G(g)$ varying isotopically from $G = G(f)$ with g, disjoint from the critical values of g, and such that $G(g) \subset g^{-1}(G(g))$.

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We say that a set of g is combinatorially bounded if $G(g)$, $U(g)$ and $n(g)$ exist as above for all g in the set, with an upper bound on the integers $n(g)$.

Theorem

Let V_1 be a maximal connected set of g for which $G(g)$, $U(g)$ and $n(g)$ exist as before, within a variety V of rational maps on which the critical values vary isotopically. Let $V_2 \subset V_1$ be such that $\overline{V_2} \setminus V_1 \neq \emptyset$, where $\overline{V_2}$ denotes the closure in V.

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Then V_2 is not combinatorially bounded.

The proof involves looking at the Hausdorff limit of $G(g_n)$ for a sequence g_n with $g_n \to \partial V_2$. Independent of combinatorial boundedness, the first step is just to show that if the g_n all lie in a bounded set in V, then the edges of $G(g_n)$ remain homotopically bounded. Once that is obtained, it is quite straightforward to show that combinatorial boundedness implies that $\lim_{n\to\infty} G(g_n)$ is a graph with the same properties as before.