Newton's method for certain entire functions

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Newton's method

Definition (Newton function)

Let f be a non-constant entire function. We $\mathcal{N}_f$ define by

$$\mathcal{N}_f := id - \frac{f}{f'}$$

and call it the Newton function of f. In order to find the zeros of f we iterate $\mathcal{N}_f$ and we call this approach the Newton's method.

The zeros of f are the attracting fixed points of $\mathcal{N}_f$.
We are interested in a set of starting points such that we find all zeros of f.
The best possible set is the set of all zeros.
We are going to consider transcendental entire functions of the form:

$$f(z) = p(z)e^z - 1.$$

Where do I come from?

Hubbard, Schleicher, Sutherland solved the problem for practical applications in a satisfying manner.

Theorem (Hubbard, Schleicher, Sutherland 2001)

For every $d \geq 2$ there is a set $\mathcal{S}_d$ containing at most $1.11d\log^2(d)$ points in $\mathbb{C}$ such that for every polynomial for every polynomial p of degree d and every of its zeros there is a point $s \in \mathcal{S}_d$ that is in the immediate basin of the chosen zero with respect to $\mathcal{N}_p$.

What can we say about transcendental entire functions instead of polynomials?

Where do I come from?

For a polynomial $p(z) = z^d + \sum_{k=0}^{d-1} a_k z^k$ of degree $d \in \mathbb{N}$, $a_k \in \mathbb{C}$, we define

$$R_p := \max\left\{ 1, \sum_{k=0}^{d-1} |a_k| \right\}.$$

R_p is the maximal modulus of any zero of p. For a positive, real R let

$$\mathcal{F}_R := \left\{ f(z) = p(z)e^z - 1 \;\middle|\; p(z) = z^d + \sum_{k=0}^{d-1} a_k z^k,\ R_P < R \right\}.$$

Bergweiler gave an answer with an extra condition.

Theorem (Bergweiler 1993)

Let $R > 0$ and $f \in \mathcal{F}_R$. If $\left(\mathcal{N}_f^{\circ n}(z)\right)_{n\in\mathbb{N}}$ converges to a finite limit for all $z \in \{ w \mid f''(w) = 0 \}$, then $\left(\mathcal{N}_f^{\circ n}(z)\right)_{n\in\mathbb{N}}$ converges for an open dense set of the complex plane.

The point I want to reach.

Theorem (S1)

Let $R > 0$. For every $d \geq 2$ there is a set $\mathcal{S}_{d,R} \subset \mathbb{C}$ such that for every function $f \in \mathcal{F}_R$ and every of its zeros there is a point $s \in \mathcal{S}_{d,R}$ which is in the immediate basin of the chosen zero with respect to $\mathcal{N}_f$.

- $\mathcal{S}_{d,R}$ is not finite. But ...
- For all but finitly many zeros we have only one starting point in $\mathcal{S}_{d,R}$.
- For the remaining zeros there are only finitly many starting points in $\mathcal{S}_{d,R}$.
- $\mathcal{S}_{d,R}$ depends only on d and R.

How do we prove this claim?

Preliminary

Definition (Accessible boundary point)

Let U be a simply connected domain in the Riemann Sphere $\widehat{\mathbb{C}}$.
A point $v \in \partial U$ is called accessible (from U),
if there exists a curve $\gamma : [0,1) \to U$ such that $\gamma(t) \to v$ for $t \to 1$.

Definition (Access to a boundary point)

For a fixed $w \in U$ and an accessible point $v \in \partial U$
we call the homotopy class of curves $\gamma : [0,1) \to U$ such that $\gamma(0) = w$
and $\gamma(t) \to v$ for $t \to 1$ an access to v from U.
We call an access H to v from U invariant,
if there exists a curve $\gamma \in H$ such that $\mathcal{N}_f(\gamma) \in H$ holds.

Preliminary

Now consider a non-linear entire function f and its Newton-function $\mathcal{N}_f$. Let w be an attracting fixed point of $\mathcal{N}_f$ and U its immediate basin.

Are there accesses to ∞ from U and how many?

Theorem (Mayer, Schleicher 2006)

The immediate basin U of w is simply connected and unbounded.

Theorem (Baranski, Fagella, Jarque, Karpinska 2014)

If $|w|$ is large, then the immediate basin U has infinity many acceses to ∞ and only one is invariant.

Let's take a look at some pictures

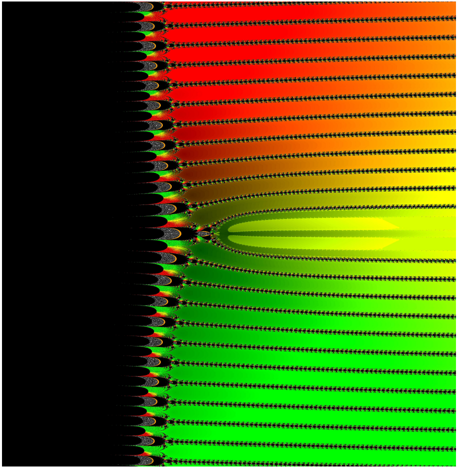

Figure : $f(z) = \left(1 - \frac{z^2}{2} + z^3\right) e^z - 1$
$(-50, 50) \times (-75, 75)$

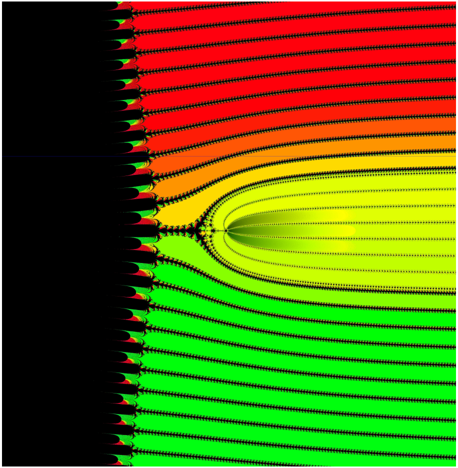

Figure : $f(z) = (z+1)^7 \, e^z - 1$
$(-75, 75) \times (-75, 75)$

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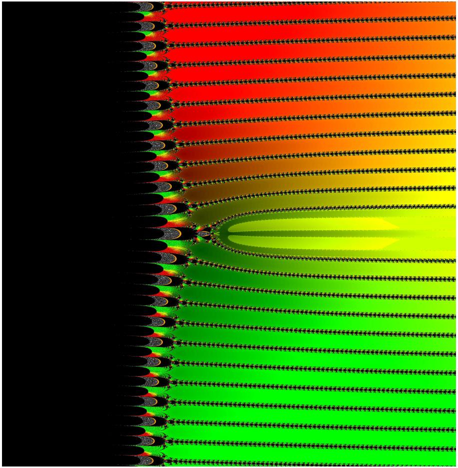

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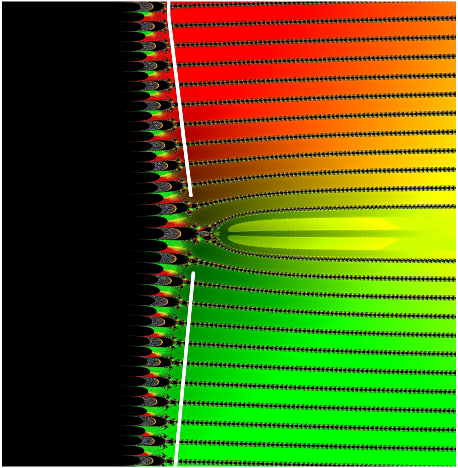

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How can we contruct the set $\mathcal{S}_{d,R}$?

(1) Approximate zeros with large modulus
(2) positions of the immediate basins

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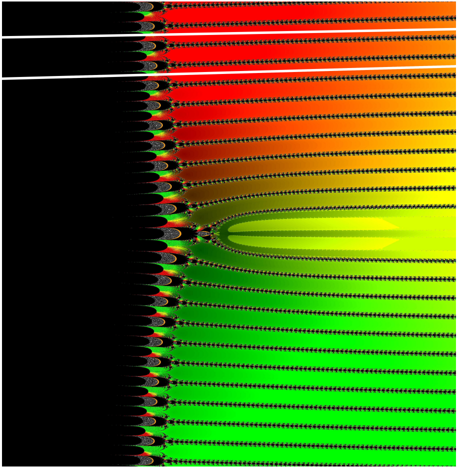

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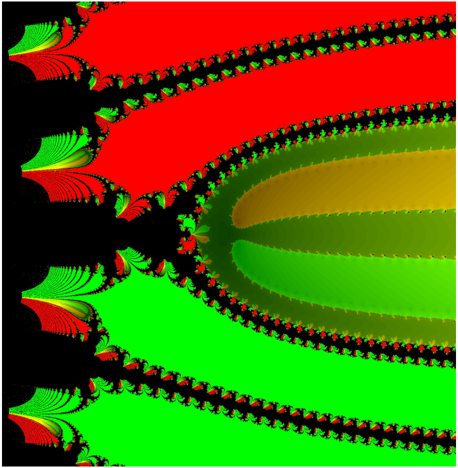

Figure : $f(z) = \left(1 - \frac{z^2}{2} + z^3\right) e^z - 1$
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How can we contruct the set $\mathcal{S}_{d,R}$?

(1) Approximate zeros with large modulus
(2) positions of the immediate basins
(3) do some magic for the smaller zeros

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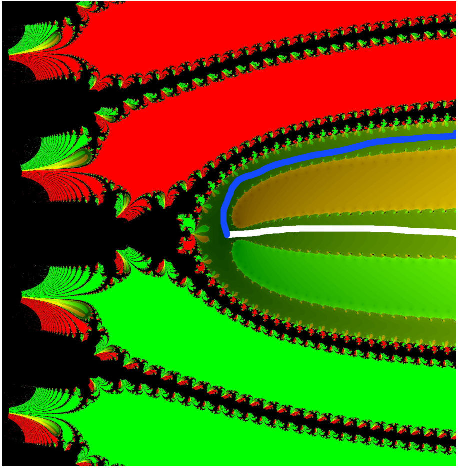

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How can we contruct the set $\mathcal{S}_{d,R}$?

(1) Approximate zeros with large modulus
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(1) Approximate zeros of large modulus

Let $w = x + iy$ denote a zero of f such that $|w|$ is large.

Idea: $p(z) \approx z^d$ for $|z|$ *large*

$$\implies \quad 0 = f(w) \approx w^d e^w - 1$$

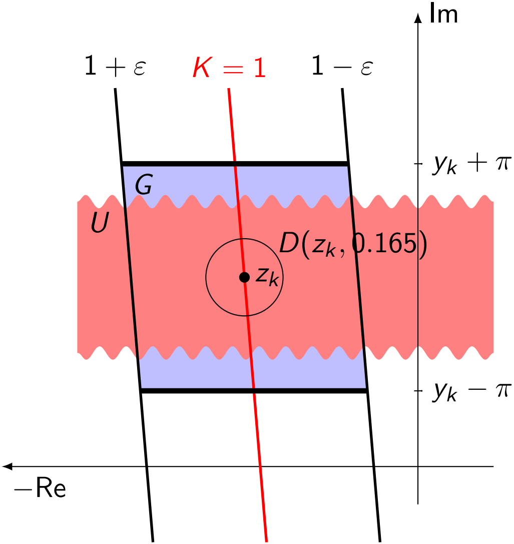
$$\implies \; x \approx -d \ln|w| = -\frac{d}{2} \ln\left(x^2 + y^2\right) \quad \wedge \quad y \approx -d \arg(w)$$

- We see that x is negativ.
- $y^2 \approx e^{\frac{-2x}{d}} - x^2$.
- This implies that y grows exponentially in $-x$.

$$\implies \; x_k := -d \ln|y_k| \quad \wedge \quad y_k := -d\frac{\pi}{2} + 2\pi k$$

We set $z_k := x_k + iy_k$ for some $k \in \mathbb{Z}$.

(1) Approximate zeros of large modulus



$$\mathcal{N}_f(z) = z - 1 + \frac{e^{-z}}{z^d} + \text{Err}$$

$$K(z) := \left|\frac{e^{-z}}{z^d}\right|$$

$|\mathcal{N}_f(z_k) - z_k| < \varepsilon$
$|\mathcal{N}_f(z) - z| \geq \varepsilon\ , \forall z \in \partial G$
Minimum principle implies the existence of a zero $w \in G$.

Some further calculations show $w \in D\,(z_k, 0.165)$.

One theorem shows $D(z_k, 0.165) \subset U$.

(2) Positions of an immediate basins

As before let w be a fixed point of $\mathcal{N}_f$ such that $|w|$ is large.

Theorem (S2)

There exists an invariant curve γ in the immediate basin U of w such that for every $x \in \mathbb{R}$ there is a point $z \in \mathrm{tr}(\gamma)$ such that $Re(z) = x$. Furthermore, this curve is contained in a horizontal strip of finite height.

Then the immediate basin of a fixed point of smaller modulus is also contained in a horizontal strip of finite height.

(2) Existence of an invariant curve

Theorem (Jankowksi 1996)

There is an unbounded curve γ in U such that

$$\gamma(0) = w\ ,\ \lim_{t\to 1} Re(\gamma(t)) = +\infty \quad \textit{and} \quad \mathcal{N}_f(\gamma) \subset \mathrm{tr}(\gamma)\,.$$

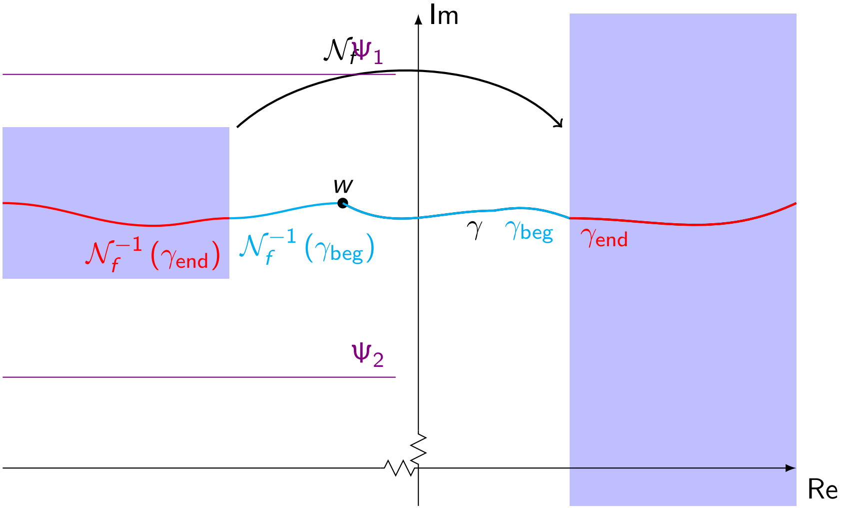
Furthermore, there is a bounded intervall $I \subset \mathbb{R}$ such that

$$\mathrm{Im}(z) \in I \textit{ for every } z \in \mathrm{Tr}(\gamma)\,.$$

In order to complete the curve, we have to find a preimage of γ in the left half plane. For this approach, we write the Newton function of f in the following way:

$$\mathcal{N}_f(z) = z - 1 + \frac{e^{-z}}{z^d} + \mathrm{Err}$$

(2) Existence of invariant curve



(3) Do some magic

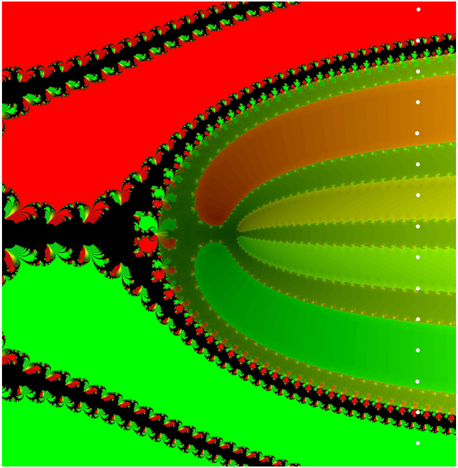


Figure : $f(z) = \left(1 - \frac{z^2}{2} + z^5\right) e^z - 1$
$(-15, 15) \times (-15, 15)$

How do we ensure that we find all zeros with small modulus?

- We use the method from Hubbard, Schleicher & Sutherland 2001.
- We showed that the immediate basins are contained in a horizontal strip of finite height.
- We have to show that the accesses to infinity have a minimal width.
- Put some equidistant points on a vertical line.

The end

Thank you for your attention.