Introduction to the dynamics of holomorphic endomorphisms of $\mathbb{P}^k$

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Outline

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Complex Projective Space $\mathbb{CP}^k$

Let $z, w \in \mathbb{C}^{k+1}$. Consider the equivalence relation:
$z \sim w$ if there is $\lambda \in \mathbb{C}^*$ such that $z = \lambda w$.

Definition

The projective space of dimension k is the quotient of $\mathbb{C}^{k+1} \setminus \{0\}$ by this relation, i.e. $\mathbb{P}^k := \mathbb{C}^{k+1} \setminus \{0\} /_\sim$.

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In other words, $\mathbb{P}^k$ is the parameter space of the complex lines passing through 0 in $\mathbb{C}^{k+1}$.

We denote by $[z_0 : z_1 : \ldots : z_k]$ the point of $\mathbb{P}^k$ associated to the point $(z_0, z_1, \ldots, z_k)$ of $\mathbb{C}^{k+1} \smallsetminus \{0\}$.

Complex Projective Space $\mathbb{CP}^k$

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We equip the space $\mathbb{P}^k$ with the *Fubini-Study* metric.

Holomorphic Endomorphisms of $\mathbb{P}^k$

Theorem

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism. Then f is described by the coordinates $[f_0 : f_1 : \ldots : f_k]$ where each f_j is a homogeneous polynomial of degree d and the f_j have no common zero except the origin.

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- $f : \mathbb{P}^k \to \mathbb{P}^k$, $[z_0 : z_1 : \ldots : z_k] \mapsto [z_0^d : z_1^d : \ldots : z_k^d]$, $d \geq 2$. So $f \in \mathcal{H}_d$.

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- $g : \mathbb{C}^2 \to \mathbb{C}^2$, $g(z_0, z_1) = (z_0 + 1, z_0^d + z_0 z_1^{d-1})$.

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- $g : \mathbb{C}^2 \to \mathbb{C}^2$, $g(z_0, z_1) = (z_0 + 1, z_0^d + z_0 z_1^{d-1})$.
  We extend g to $\mathbb{P}^2$:
  $\tilde{g} : [z_0 : z_1 : z_2] \mapsto [z_0 z_2^{d-1} + z_2^d : z_0^d + z_0 z_1^{d-1} : z_2^d]$. Then $\tilde{g} \notin \mathcal{H}_d$.

Fatou and Julia sets

As in dynamics in one complex variable, we define:

Definition

Let $f \in \mathcal{H}_d(\mathbb{P}^k)$. We define the Fatou set $\mathcal{F}$ of f as the largest open set where the family of iterates $\{f^n\}_{n=1,2,...}$ is locally equicontinuous. The Julia set $\mathcal{J}$ of f is defined by $\mathcal{J} := \mathbb{P}^k \setminus \mathcal{F}$.

Outline

Pluriharmonic Functions

Definition

Let $\Omega \subset \mathbb{C}^n$ be an open subset and $u \in \mathcal{C}^2(\Omega)$ be a real valued function.

- u is said to be *pluriharmonic* in Ω if, for every $a, b \in \mathbb{C}^n$, the function $\lambda \mapsto u(a + \lambda b)$ is harmonic in $\{\lambda \in \mathbb{C} | a + \lambda b \in \Omega\}$.
- u is *pluriharmonic* in Ω if

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0 \quad \text{in } \Omega, \text{ where } j, k = 1, \ldots, n.$$

Plurisubharmonic Functions

Definition

Let Ω be an open subset of $\mathbb{C}^n$, and let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω.
The function u is said to be *plurisubharmonic* if for each $a \in \Omega$, $b \in \mathbb{C}^n$, the function $\lambda \mapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} | a + \lambda b \in \Omega\}$.

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Example:

If $f : U \to \mathbb{C}$ is holomorphic in the open set $U \subset \mathbb{C}^n$ and $f \not\equiv 0$, then the function $\log |f|$ is plurisubharmonic in U.

Differential Forms and Currents

Differential Forms and Currents

- $\mathcal{D}^{p,q}(\Omega)$: the space of differential forms of class $\mathcal{C}^\infty$ in $\Omega \subset \mathbb{C}^n$ with compact support and whose bidegree is (p,q).
- If $\varphi \in \mathcal{D}^{p,q}(\Omega)$, then $\varphi = \sum \varphi_{IJ} dz_I \wedge d\bar{z}_J$, where $\varphi_{IJ} \in \mathcal{C}_k^\infty(\Omega)$ and $\sharp I = p, \sharp J = q$.

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Definition

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The elements of the dual space $(\mathcal{D}^{n-p,n-q}(\Omega))'$ are called *currents* of bidegree (p,q).

- A current S is written as: $S = \sum S_{IJ} dz_I \wedge d\bar{z}_J$, where the coefficients S_{IJ} are distributions.
- If S is a positive $(p,p)-$current, then the coefficients S_{IJ} are measures.

Currents and Plurisubharmonic Functions

- We define the diff. operator $dd^c = 2i\partial\overline{\partial}$.
- A function $u \in L^1_{loc}(\Omega)$ is a.e. equal to a p.s.h. function iff

$$dd^c u = 2i \sum_{i,j} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \geq 0.$$

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Important Theorem

Every $(1,1)$ positive closed current S corresponds to a plurisubharmonic function u. The function u verifies the equation

$$dd^c u = S.$$

Example

- We consider the *Fubini-Study* differential form ω_{FS} in $\mathbb{P}^k$.
- ω_{FS} is written locally as

$$\omega_0 = dd^c \log |z|.$$

Outline

Green Current

Theorem

If $f \in \mathcal{H}_d(\mathbb{P}^k)$, then the sequence of currents

$$\left\{ \frac{1}{d^n}(f^n)^*(\omega_{FS}) \right\}_{n \in \mathbb{N}}$$

converges to a $(1,1)$ positive closed current T, the *Green current*.

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- $G = \lim_{n\to\infty} G_n = \sup\{\log |z|, \log |w|, \log |t|\}$.
- The plurisubharmonic function G corresponds to the Green current of f:

$$dd^c G = T.$$

Julia set & Green current

Theorem

If $f \in \mathcal{H}_d$ and let T be the Green current associated to f, then

$$Supp\ T = \overline{\{T \neq 0\}} = \mathcal{J}.$$

The Green measure

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- The Green measure is invariant by f, i.e. $f_*\mu = \mu$.

Theorem

The Green measure μ is the only invariant measure that maximises the entropy.

Thank you for your attention.