On a quasiregular map with non-escaping set of finite measure

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London, 12 March 2015

1. Why quasiregular maps?

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2. Construction of qr sine and results

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We can easily generalise this definition to higher dimensions. But if we take $G \subset \mathbb{R}^d$ open and $f : G \to \mathbb{R}^d$ satisfying

- f is C^1 in the real sense and
- $\|Df(x)\|^d = J_f(x)$ for all $x \in G$,

then f is either constant or a sense preserving Möbius transformation.

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- there exists $K \geq 1$, such that $\|Df(x)\|^d \leq KJ_f(x)$ a.e.,

where $W^1_{d,loc}(\mathbb{R}^d)$ denotes the set of all functions $f = (f_1, \dots f_d) : U \to \mathbb{R}^d$, for which the weak partial first order derivatives $\partial_k f_i$ exist and are locally in L^d.

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Hence f maps infinitesimal small balls to infinitesimal small ellipsoids with bounded eccentricity.

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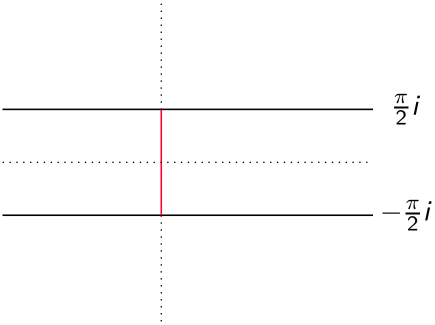
- There is no obvious definition of the Julia set of non-uniformly quasiregular maps.
- However, the escaping set

$$I(f) := \left\{ x \in \mathbb{R}^d : \|f^n(x)\| \to \infty \text{ for } n \to \infty \right\}$$

is still easy to define.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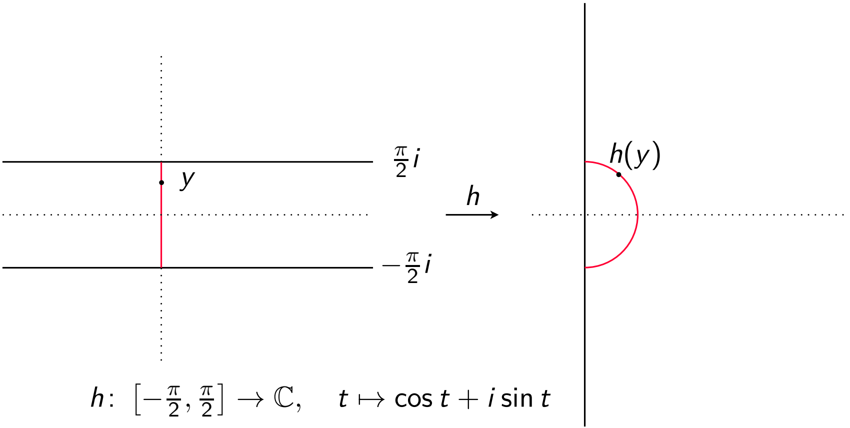

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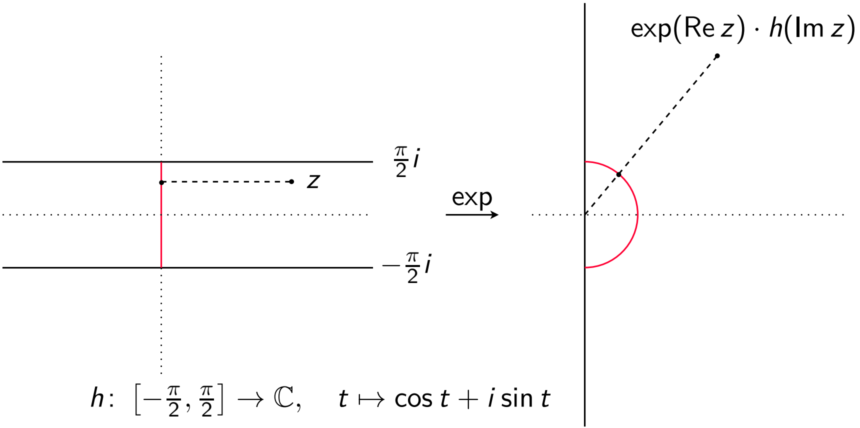

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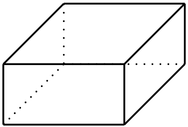

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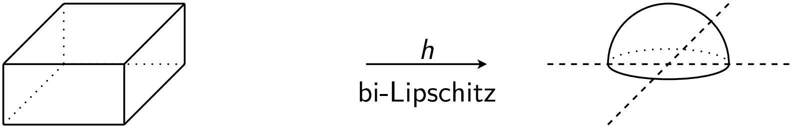

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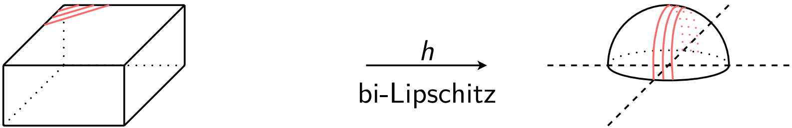

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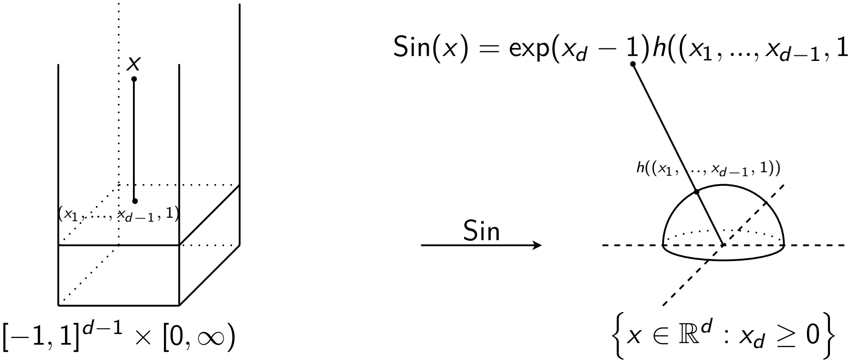

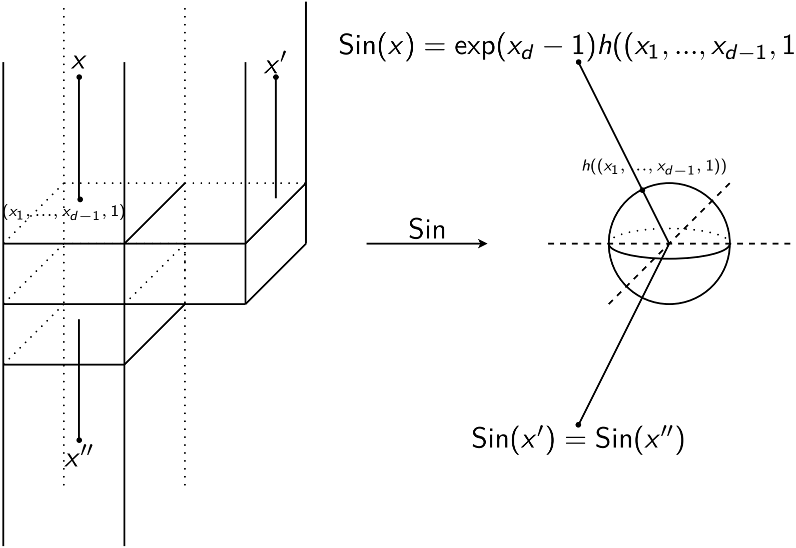

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For λ sufficiently large, the periodic points of $f := \lambda \operatorname{Sin}$ are dense in $\mathbb{R}^d$. Furthermore, f has the blowing-up property everywhere in $\mathbb{R}^d$, that is

$$\bigcup_{k=0}^{\infty} f^k(U) = \mathbb{R}^d, \quad \text{for any non-empty open set } U \subset \mathbb{R}^d.$$

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The escaping set of Sin has positive measure, i.e.

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$$S := \mathbb{R}^d \setminus L.$$

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For $x \in \mathbb{R}^d$, x_d large, there exists a (rapidly) decreasing, positive sequence $(\Delta_n(x_d))$, such that

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$$\mathrm{dens}(\mathcal{T}, \mathcal{T}_0 \cap Q(x) \ge \prod_{n=1}^{\infty} (1 - \Delta_n(x_d)) > 0$$

and thus $\mathrm{meas}(\mathcal{T}) > 0$.

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In the case of the quasiregular analogue of sine we get the following analogue of this result:

Theorem 2

Let Tr be a tract of Sin. Then $\text{Tr} \setminus I(\text{Sin})$ has finite measure.

To prove this, we show that the initial tract Tr minus the set $\mathcal{T}$ has finite measure.

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Lemma

Let $x \in \mathbb{R}^d$ with x_d large. Then

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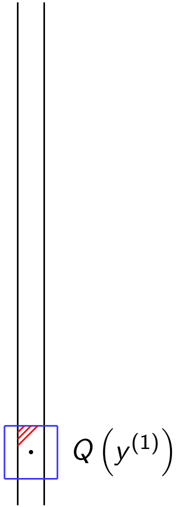
Now we cover the initial tract with cubes $Q\left(y^{(j)}\right)$, $y^{(j)} \in \mathbb{R}^d$, in the following way: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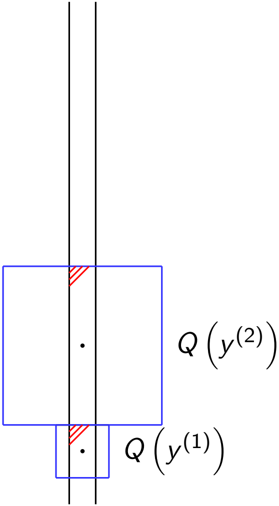
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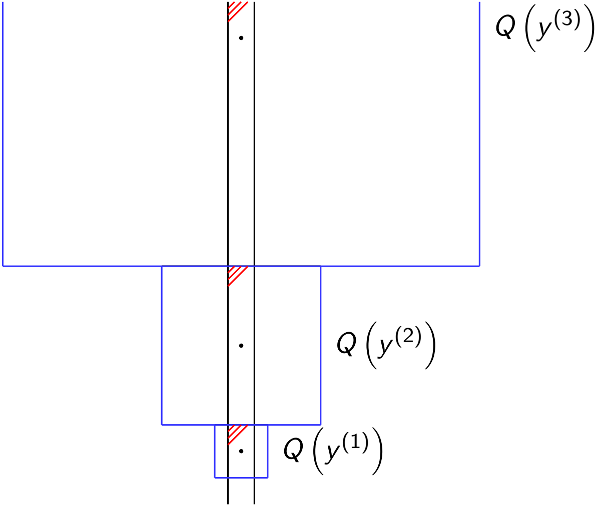
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Mayer uses this approach to construct quasiregular power mappings based on Zorich maps.

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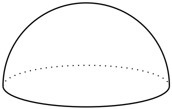
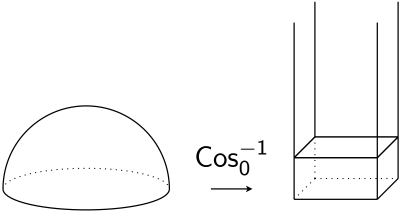
Theorem 1 and 2 hold for $\lambda\,\mathrm{Cos}$, $\lambda > 0$ with the same proofs.





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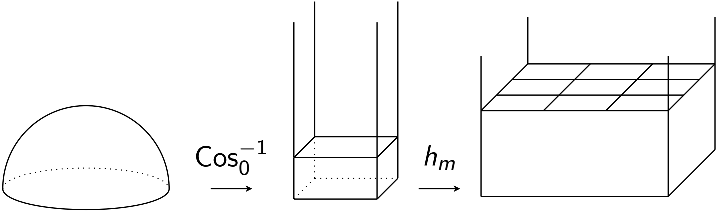

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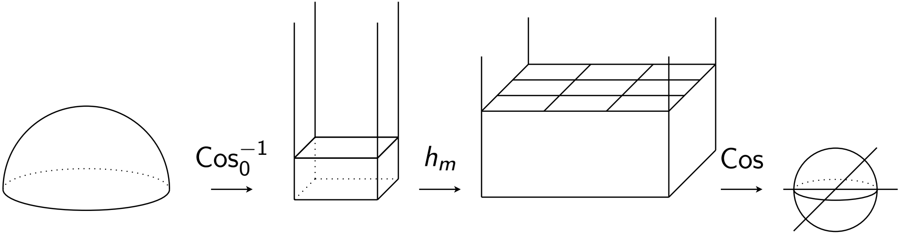
$$h_m\colon [0,2]^{d-1} \times [0,\infty) \to [0,2m]^{d-1} \times [0,\infty), \quad h_m(x) = mx$$



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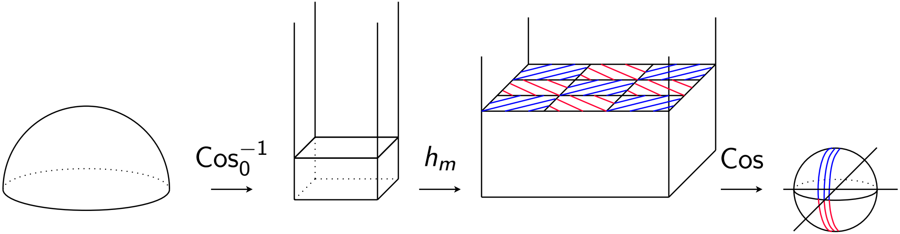

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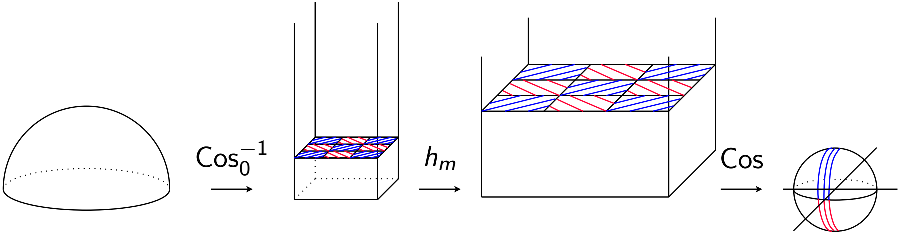

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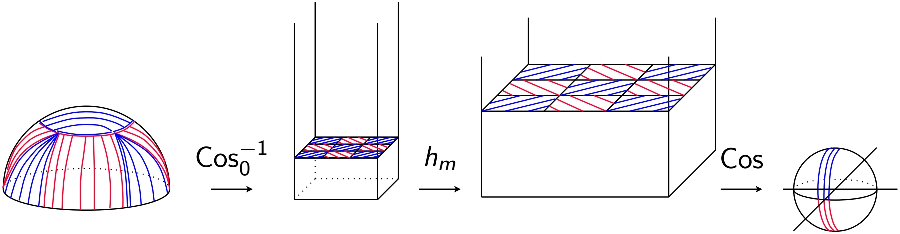

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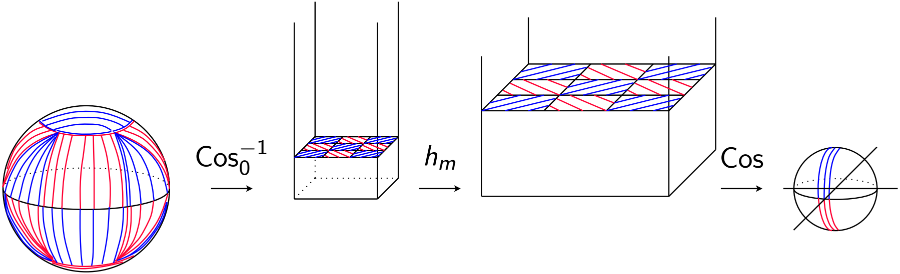

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We call

$$P_m \colon \mathbb{R}^d \to \mathbb{R}^d, \quad P_m(x) = \begin{cases} \operatorname{Cos} \circ h_m \circ \operatorname{Cos}_0^{-1}(x) \text{ for } x \in \mathbb{H}^+ \\ \overline{\operatorname{Cos} \circ h_m \circ \operatorname{Cos}_0^{-1}(\overline{x})} \text{ for } x \in \mathbb{H}^- \end{cases}$$

quasiregular power mapping, where

$$\overline{x} = \overline{(x_1, ..., x_d)} = (x_1, ..., x_{d-1}, -x_d)$$

denotes the reflection at the hyperplane $\mathbb{R}^{d-1} \times \{0\}$.

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Theorem 3

For $m \geq d + 1$ we get, that $\operatorname{meas}(I^c(\operatorname{Cos} \circ P_m)) < \infty$.

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Theorem 3

For $m \geq d+1$ we get, that $\operatorname{meas}(I^c(\operatorname{Cos} \circ P_m)) < \infty$.

For $m \leq d$ we get $\operatorname{meas}(I^c(\operatorname{Cos} \circ P_m)) = \infty$ or $\operatorname{meas}(I^c(\operatorname{Cos} \circ P_m)) = 0$, depending on the initial bi-Lipschitz map h.

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow[\mathrm{Cos} \circ h_m]{} & \mathbb{R}^d \\ {\scriptstyle h_m} \downarrow & & \downarrow {\scriptstyle h_m} \\ \mathbb{R}^d & \xrightarrow[h_m \circ \mathrm{Cos}]{} & \mathbb{R}^d \end{array}$$

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{\text{Cos}\circ P_m} & \mathbb{R}^d \\ P_m \downarrow & & \downarrow P_m \\ \mathbb{R}^d & \xrightarrow[\text{Cos}\circ h_m]{} & \mathbb{R}^d \\ h_m \downarrow & & \downarrow h_m \\ \mathbb{R}^d & \xrightarrow[h_m\circ \text{Cos}]{} & \mathbb{R}^d \end{array}$$

$$\begin{array}{ccc}
\mathbb{R}^d & \xrightarrow{\operatorname{Cos}\circ P_m} & \mathbb{R}^d \\
\Big\downarrow P_m & & \Big\downarrow P_m \\
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\end{array}$$

Show that $P_m^{-1}(I^c(\operatorname{Cos}\circ h_m))$ has finite measure by using theorem 2 and the fact, that P_m has degree m^{d-1}.

Thank you very much for your attention!