Gaussian Integer Continued Fractions, the Picard Group, and Hyperbolic Geometry

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Compositions of Möbius Transformations

A *Möbius transformation* is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

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where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

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Definition

Given a set $\mathcal{F}$ of Möbius transformations, we define a *composition sequence* drawn from $\mathcal{F}$ to be a sequence of Möbius transformations F_n such that

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n$$

where each $f_i \in \mathcal{F}$.

Note the order of composition.

Picard Composition Sequences

Let $\mathcal{F}$ denote the set of all Möbius transformations

$$f_a(z) = \frac{az+1}{z} = a + \frac{1}{z},$$

where $a \in \mathbb{Z}[i]$, that is, a is a Gaussian integer.

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Definition

We define a *Picard composition sequence* to be a composition sequence drawn from $\mathcal{F}$.

The functions f_a generate the Picard group, G, the group of Möbius transformations

$$f(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{Z}[i]$ and $|ad - bc| = 1$. So each F_n lies in G. This group will be important later.

Continued Fractions

Notice that

$$\begin{aligned} F_n(z) &= f_{a_1} \circ f_{a_2} \circ f_{a_3} \circ \dots f_{a_n}(z) \\ &= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n + \cfrac{1}{z}}}}, \end{aligned}$$

Continued Fractions

Notice that

$$F_n(z) = f_{a_1} \circ f_{a_2} \circ f_{a_3} \circ \dots f_{a_n}(z)$$
$$= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n + \cfrac{1}{z}}}},$$

so the values $F_n(\infty)$ are the convergents of some continued fraction with entries equal to 1 ‘along the top’ and Gaussian integers ‘along the bottom’.

Gaussian Integer Continued Fractions

Definition

A finite *Gaussian integer continued fraction* is a continued fraction of the form

$$[a_1, a_2, a_3, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}},$$

where $a_i \in \mathbb{Z}[i]$ for $i = 1, 2, \ldots, n$.

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where $a_i \in \mathbb{Z}[i]$ for $i = 1, 2, \ldots, n$.

An infinite Gaussian integer continued fraction is defined to be the limit

$$[a_1, a_2, \ldots] = \lim_{i \to \infty} [a_1, a_2, \ldots, a_i],$$

of its sequence of convergents.

Convergence of Gaussian Integer Continued Fractions

The question

“When does a Picard composition sequence $F_n = f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n}$ converge at ∞?"

can be reformulated as the question

“When does a Gaussian integer continued fraction $[a_1, a_2, \ldots, a_n]$ converge?"

Literature on this topic generally restricts to certain classes of Gaussian integer continued fractions, such as those obtained using algorithms. See, for example, Dani and Nogueira [2].

Question: Can we find a more general condition for convergence that can be applied to all Gaussian integer continued fractions?

The Geometry of the Picard Group

Recall that the elements F_n of a Picard composition sequence are elements of the Picard group, G, which is a group of conformal automorphisms of $\hat{\mathbb{C}}$.

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In fact, G is a Kleinian group - a discrete group of isometries of the hyperbolic upper half-space $\mathbb{H}^3$. This allows us to form the Picard-Farey graph.

The Picard-Farey Graph

Definition

The *Picard-Farey graph*, $\mathcal{G}$, is formed as the orbit of the vertical line segment L with endpoints 0 and ∞ under the Picard group.

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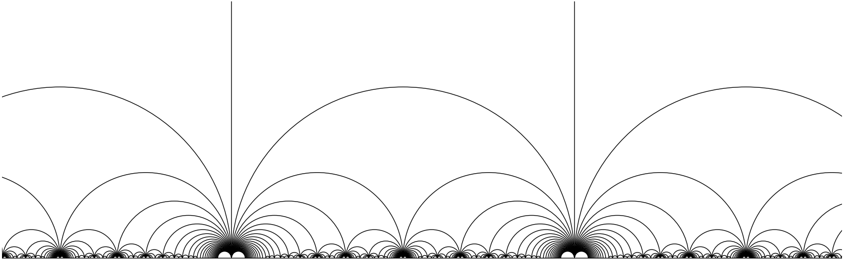
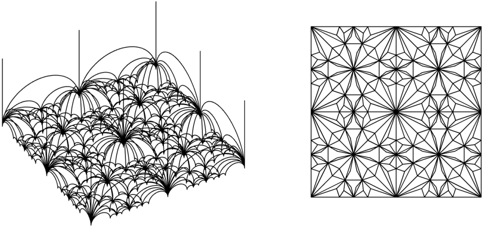
The *Picard-Farey graph*, $\mathcal{G}$, is formed as the orbit of the vertical line segment L with endpoints 0 and ∞ under the Picard group.

It is a three-dimensional analogue of the Farey graph.



Properties of the Picard-Farey graph



- The Picard-Farey graph is the 1-skeleton of a tessellation of $\mathbb{H}^3$ by ideal hyperbolic octahedra.
- The vertices $V(\mathcal{G})$ are of the form $\frac{a}{c}$ with $a, c \in \mathbb{Z}[i]$: they are precisely those complex numbers with rational real and complex parts, and ∞ itself.
- The edges of $\mathcal{G}$ are hyperbolic geodesics. Two vertices $\frac{a}{c}$ and $\frac{b}{d}$ are neighbours - joined by an edge - in $\mathcal{G}$ if and only if $|ad - bc| = 1$.
- Elements of G are graph automorphisms of $\mathcal{G}$.

Gaussian Integer Continued Fractions

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$$[a_1, a_2, \dots, a_n] = F_n(\infty).$$

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Notice that

$$F_n(0) = F_{n-1}(f_{a_n}(0)) = F_{n-1}\left(a_n - \frac{1}{0}\right) = F_{n-1}(\infty),$$

so $F_{n-1}(\infty)$ and $F_n(\infty)$ are neighbours in $\mathcal{G}$.

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so $F_{n-1}(\infty)$ and $F_n(\infty)$ are neighbours in $\mathcal{G}$.

Theorem

A sequence of vertices $\infty = v_1, v_2, \ldots, v_n = x$ forms a path in $\mathcal{G}$ if and only if it consists of the consecutive convergents of a Gaussian integer continued fraction expansion of x.

Reformulating the Theory of Gaussian Integer Continued Fractions

The question

"When does a Picard composition sequence $F_n = f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n}$ converge?"

can be reformulated as the question

"When does a Gaussian integer continued fraction $[a_1, a_2, a_3, \ldots, a_n]$ converge?"

which can be reformulated as the question

"When does a path in $\mathcal{G}$ with initial vertex ∞ converge?"

Integer Continued Fractions

Definition

A finite *integer continued fraction* is a continued fraction of the form

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}},$$

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$$[a_1, a_2, \ldots] = \lim_{i \to \infty} [a_1, a_2, \ldots, a_i],$$

of its sequence of convergents.

Some Known Theorems

Theorem

A continued fraction $[a_1, a_2, \ldots]$*, with* $a_i \in \mathbb{R}$ *and* $a_i > 0$ *for* $i > 1$*, converges if and only if* $\sum_{i=1}^{\infty} a_i$ *diverges.*

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Theorem (Śleszyński-Pringsheim)

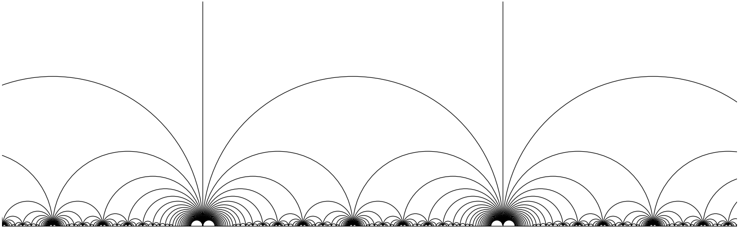
If $a_i, b_i \in \mathbb{R}$ *with* $|b_{n+1}| > |a_n| + 1$ *for all* n, *then*

$$b_1 + \cfrac{a_1}{b_2 + \cfrac{a_2}{b_3 + \ldots}}$$

converges.

The Farey Graph

Let L' denote the line segment joining 0 to ∞ in $\mathbb{H}^2$. The Farey graph, $\mathcal{H}$, is the orbit of L' under the Modular group.



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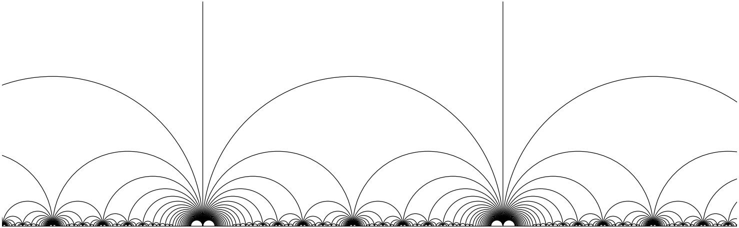
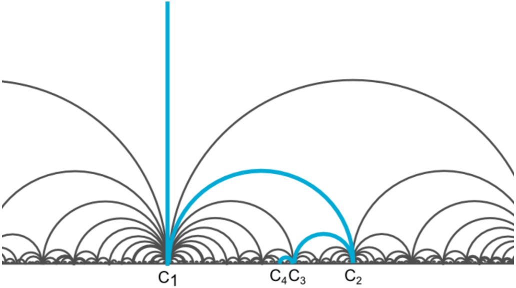


Theorem (Beardon, Hockman, Short [1])

A sequence of vertices $\infty = v_1, v_2, \ldots, v_n = x$ forms a path in $\mathcal{H}$ if and only if it consists of the consecutive convergents of a Gaussian integer continued fraction expansion of x.

Paths in the Farey graph

Take, for example, $[0, 2, 1, -3, \ldots]$



$$C_1 = 0, \quad C_2 = \frac{1}{2}, \quad C_3 = \frac{1}{3}, \quad C_4 = \frac{2}{7}, \ldots$$

Convergence of Integer Continued Fractions

Theorem

An infinite path in $\mathcal{H}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to an irrational number x if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence.

Proof.

$\Longrightarrow$ Clear.

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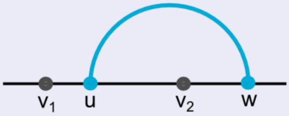
Proof.

$\Longrightarrow$ Clear. $\Longleftarrow$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, v_1 and v_2. There is some edge of $\mathcal{H}$, with endpoints u and w 'separating' v_1 and v_2.



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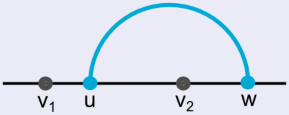

Thus the path must pass through one of u or v infinitely many times, and has a convergent subsequence. □

A Problem

The key property used here is that removing any edge of $\mathcal{H}$ separates it into two connected components.

In the Picard-Farey graph, $\mathcal{G}$, there is no such property: removing any finite number of edges will not separate $\mathcal{G}$ into two connected components.

A Problem

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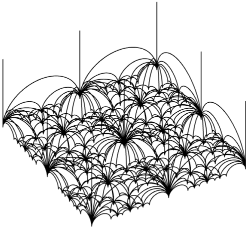
In the Picard-Farey graph, $\mathcal{G}$, there is no such property: removing any finite number of edges will not separate $\mathcal{G}$ into two connected components. Is there a 'nice' infinite set that we can use instead?

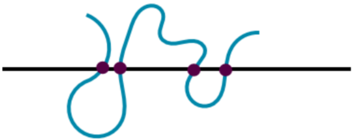
Along $\hat{\mathbb{R}}$ lies a vertical Farey graph.



Removing it separates $\mathcal{G}$ into two connected components.

The Real Line

Thus any path that crosses $\hat{\mathbb{R}}$ must pass through a vertex lying on the real line.



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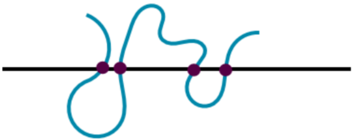


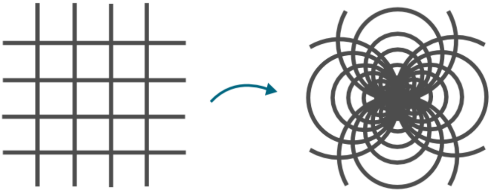
Elements of the Picard group are automorphisms of $\mathcal{G}$, so any image of $\hat{\mathbb{R}}$ has this same property.

Definition

A *Farey section* is an image of $\hat{\mathbb{R}}$ under an element of G.

Farey Sections

Farey sections cover $\hat{\mathbb{C}}$ densely.



Each Farey section has the property that if a path crosses it then it must pass through it.

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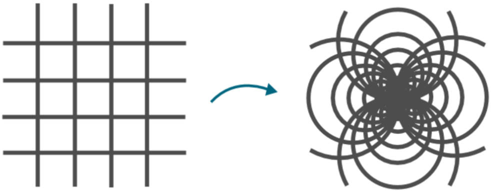

Each Farey section has the property that if a path crosses it then it must pass through it.

If a path crosses a Farey section infinitely many times, then it either has an accumulation point in that Farey section, or passes through some vertex of that Farey section infinitely many times.

Convergence of Gaussian Integer Continued Fractions

Theorem

An infinite path in $\mathcal{G}$ *with vertices* $\infty = v_1, v_2, v_3, \dots$ *converges to* $x \notin V(\mathcal{G})$ *if and only if the sequence* $v_1, v_2, \dots$ *contains no constant subsequence and has only finitely many accumulation points.*

Proof.

$\implies$ Clear.

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Proof.

$\Longrightarrow$ Clear. $\Longleftarrow$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, v_1 and v_2. There is an infinite family of Farey sections 'separating' v_1 and v_2.

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Proof.

$\Longrightarrow$ Clear. $\Longleftarrow$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, v_1 and v_2. There is an infinite family of Farey sections 'separating' v_1 and v_2.

v_i either has an accumulation point on each Farey section, or has a constant subsequence. □

Examples

Do we need the added condition? Can we say that an infinite path in $\mathcal{G}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to $x \notin V(\mathcal{G})$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence?

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Lemma

There exist paths with no constant subsequence that do not converge.

Proof.

Given $z \neq w$, choose sequences $z_i \to z$ and $w_i \to w$. Because removing finitely many edges does not disconnect $\mathcal{G}$, we can construct a simple path that passes through each z_i and w_i, and thus has both z and w as accumulation points. □

Summary

To summarise:

- Picard composition sequences can be viewed as Gaussian integer continued fractions, which can in turn be viewed as paths in the Picard-Farey graph.

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- This technique allows us to find and prove a simple condition for the convergence of Gaussian integer continued fractions, and hence Picard composition sequences.

Summary

To summarise:

- Picard composition sequences can be viewed as Gaussian integer continued fractions, which can in turn be viewed as paths in the Picard-Farey graph.
- This technique allows us to find and prove a simple condition for the convergence of Gaussian integer continued fractions, and hence Picard composition sequences.

Where next?

- What else can we say about Gaussian integer continued fractions using the Picard-Farey graph?

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- This technique allows us to find and prove a simple condition for the convergence of Gaussian integer continued fractions, and hence Picard composition sequences.

Where next?

- What else can we say about Gaussian integer continued fractions using the Picard-Farey graph?
- Can we use hyperbolic geometry to study the continued fractions associated to other types of composition sequences?

Thanks for listening!

:)

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