Escaping Fatou components of transcendental self-maps of the punctured plane

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Sketch of the talk

1. Introduction to holomorphic self-maps of \mathbb{C}^\ast

- 2. The escaping set
- 3. Preliminaries on approximation theory
- Sketch of the constructions of wandering domains and Baker domains

Transcendental self-maps of \mathbb{C}^*

Let $f : S \subseteq \widehat{\mathbb{C}} \to S$ be holomorphic s.t. $\widehat{\mathbb{C}} \setminus S$ are essential singularities. By Picard's theorem there are three interesting cases:

- $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the **Riemann sphere** (rational functions);
- $S = \mathbb{C}$, the **complex plane** (transcendental entire functions);
- $S = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the punctured plane.

Råd53 H. Rådström, *On the iteration of analytic functions*, Math. Scand. 1 (1953), 85–92. Bha69 P. Bhattacharyya, *Iteration of analytic functions*, PhD Thesis (1969), University of London, 1969.

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Holomorphic self-maps of \mathbb{C}^* were first studied in 1953 by Rådström.

Theorem (Bhattacharyya 1969)

Every transcendental function $f:\mathbb{C}^*\to\mathbb{C}^*$ is of the form

$$f(z) = z^n \exp(g(z) + h(1/z))$$

for some $n \in \mathbb{Z}$ and g, h non-constant entire functions.

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The escaping set of a transcendental entire function

The escaping set of a transcendental entire function f,

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

was introduced by Eremenko in 1989.

Theorem (Eremenko 1989, Eremenko & Lyubich 1992)

Let f be a transcendental entire function. Then,

1.
$$I(f) \cap J(f) \neq \emptyset;$$

$$12. \ J(f) = \partial I(f);$$

- 13. all the components of $\overline{I(f)}$ are unbounded;
- 14. if $f \in \mathcal{B}$, then $I(f) \subseteq J(f)$.

Here ${\mathcal B}$ denotes the so-called Eremenko-Lyubich class:

 $\mathcal{B} := \{ f \text{ transcendental entire function } : \operatorname{sing}(f^{-1}) \text{ is bounded} \}.$

EL92 A. Eremenko, and M. Lyubich *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020.

Ere89 A. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Banach Center Publ. 23 (1989), 339-345.

The escaping set of a transcendental self-map of \mathbb{C}^*

If f is a transcendental self-map of \mathbb{C}^* , the **escaping set** of f is $I(f) := \{z \in \mathbb{C}^* : \omega(z, f) \subseteq \{0, \infty\}\},$ where $\omega(z, f) := \bigcap_{n \in \mathbb{N}} \overline{\{f^k(z) : k \ge n\}}.$ Then I(f) contains $I_0(f) := \{z \in \mathbb{C}^* : f^n(z) \to 0 \text{ as } n \to \infty\},$ $I_{\infty}(f) := \{z \in \mathbb{C}^* : f^n(z) \to \infty \text{ as } n \to \infty\}.$

Mar14 D. Martí-Pete, The escaping set of transcendental self-maps of the punctured plane, arXiv:1412.1032, December 2014.

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where $\omega(z, f) := \bigcap_{n \in \mathbb{N}} \overline{\{f^k(z) : k \ge n\}}$. Then I(f) contains

$$\begin{split} I_0(f) &:= \{ z \in \mathbb{C}^* : f^n(z) \to 0 \text{ as } n \to \infty \} \,, \\ I_\infty(f) &:= \{ z \in \mathbb{C}^* : f^n(z) \to \infty \text{ as } n \to \infty \} \,. \end{split}$$

We define the **essential itinerary** of a point $z \in I(f)$ to be the sequence $e = (e_n) \in \{0, \infty\}^{\mathbb{N}}$ such that

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leq 1, \\ \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

for all $n \ge 0$. The set of points whose essential itinerary is eventually a shift of e is

$$I_e(f) := \{z \in I(f) : \exists \ell, k \in \mathbb{N}, \forall n \ge 0, |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\}.$$

Mar14 D. Martí-Pete, The escaping set of transcendental self-maps of the punctured plane, arXiv:1412.1032, December 2014.

Eremenko's properties

Theorem (Martí-Pete 2014)

Let f be a transcendental self-map of \mathbb{C}^* . For each $e \in \{0,\infty\}^{\mathbb{N}}$,

- $11. I_e(f) \cap J(f) \neq \emptyset,$
- 12. $J(f) = \partial I_e(f)$, and $J(f) = \partial I(f)$,
- 13. the connected components of $\overline{I_e(f)}$ are unbounded, and hence the connected components of $\overline{I(f)}$ are unbounded,

$$\text{ I4. if } f \in \mathcal{B}^* \text{, then } I(f) \subseteq J(f).$$

The analog of class \mathcal{B} in \mathbb{C}^* is

 $\mathcal{B}^* := \{f \text{ transc. self-map of } \mathbb{C}^* : \operatorname{sing}(f^{-1}) \text{ is bounded away from } 0, \infty\}.$

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Escaping Fatou components

Let U be a Fatou component of a transcendental function f:

- U is a wandering domain of f if $f^m(U) = f^n(U)$ implies m = n.
- U is a **Baker domain** of period p of f if ∂U contains an essential singularity α and $f_{III}^{np} \rightrightarrows \alpha$ as $n \to \infty$. Cowen classified them into three kinds according to whether f_{III}^{p} is eventually conjugated to
 - ► $z \mapsto \lambda z$, $\lambda > 1$, on $\mathbb{H} \longrightarrow U$ is a hyperbolic Baker domain,

 - ▶ $z \mapsto z \pm i$, on \mathbb{H} $\rightsquigarrow U$ is a simply parabolic Baker domain,
 - $z \mapsto z + 1$, on \mathbb{C}
- $\rightarrow U$ is a **doubly parabolic** Baker domain.

Cow81 C.C. Cowen, Iteration and the solution of functional equations for functions analytic in the unit disk, Trans. Am. Math. Soc. 265 (1981), 69-95.

Bak87 I.N. Baker, Wandering domains for maps of the punctured plane, Ann. Acad. Sci. Fenn. Ser. A I Math. 12 (1987), no. 2, 191-198.

Kot90 J. Kotus, The domains of normality of holomorphic self-maps of \mathbb{C}^* , Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990), no. 2, 329-340.

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Baker, Mukhamedshin and Kotus used approximation theory to construct transcendental self-maps of \mathbb{C}^* with wandering domains $(e = \overline{0}, \overline{\infty}, \overline{0\infty})$ and Baker domains $(e = \overline{0}, \overline{\infty})$.

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Q: Are there escaping Fatou components with any essential itinerary?

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Example 1: a wandering domain

The function $f(z) = z \exp\left(\frac{\sin z}{z} + \frac{2\pi}{z}\right)$ is a transcendental self-map of \mathbb{C}^* which has a bounded wandering domain escaping to ∞ . Note that

$$f(z) = z + \sin z + 2\pi + o(1)$$
 as $\operatorname{Re} z \to \infty$.



Example 2: hyperbolic Baker domains

For every $\lambda > 1$, the function $f_{\lambda}(z) = \lambda z \exp(e^{-z} + 1/z)$ is a transcendental self-map of \mathbb{C}^* which has a hyperbolic Baker domain escaping to ∞ . Note that

 $f(z) \sim \lambda z$ as $\operatorname{Re} z \to \infty$.



Example 3: a doubly parabolic Baker domain

The function $f(z) = z \exp((e^{-z} + 1)/z)$ is a transcendental self-map of \mathbb{C}^* which has a simply parabolic Baker domain escaping to ∞ . Note that

$$f(z)=z+1+o(1)$$
 as $\operatorname{\mathsf{Re}} z o\infty.$



Theorem (Martí-Pete)

For each $e \in \{0,\infty\}^{\mathbb{N}}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map of \mathbb{C}^* , f, such that $\operatorname{ind}(f) = n$ and $I_e(f)$ contains a wandering domain.

Theorem (Martí-Pete)

For each periodic $e \in \{0,\infty\}^{\mathbb{N}}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map of \mathbb{C}^* , f, such that $\operatorname{ind}(f) = n$ and $I_e(f)$ contains a Baker domain, which can be taken to be hyperbolic, simply parabolic or doubly parabolic.

Remark: We can also construct *entire functions* with no zeros having escaping Fatou components (to ∞) of each of these kinds.

Mar D. Martí Pete, Escaping Fatou components of transcendental self-maps of the punctured plane, in preparation.

Preliminaries on approximation theory

For $0 < \alpha \leq 2\pi$, define the *sector* of angle α

 $W_{\alpha} := \{ z \in \mathbb{C} : | \arg z | \leqslant \alpha/2 \}.$

Theorem (Arakeljan 1964)

Suppose $F \subseteq \mathbb{C}$ is a closed set such that $\widehat{\mathbb{C}} \setminus F$ is connected and locally connected at ∞ and F lies in a sector W_{α} for some $0 < \alpha \leq 2\pi$. Suppose $\varepsilon(t)$ is a real function, continuous and positive for $t \ge 0$, such that

$$\int_{1}^{+\infty} t^{-(\pi/lpha)-1}\logarepsilon(t)dt>-\infty.$$

Then every function $f:F\to\mathbb{C}$ holomorphic on int F and continous on F, there is an entire function g such that

$$|f(z) - g(z)| < \varepsilon(|z|), \text{ for all } z \in F.$$

Hence, if $F \subseteq W_{\alpha}$ with $\alpha < \pi$ we can take $\varepsilon(|z|) = O(e^{-\gamma|z|})$ with $\gamma > 0$.

Ara64 N. U. Arakeljan, Uniform and asymptotic approximation by entire functions on unbounded closed sets (Russian), Dokl. Akad. Nauk SSSR 157 (1964), 9–11.

Gai87 D. Gaier, *Lectures on complex approximation*, translated from the German by Renate McLaughlin. Birkhäuser Boston, Inc., Boston, MA, 1987.

Sketch of the wandering domain construction

We use Arakeljan's theorem to construct two entire functions g and h such that $f(z) = z^n \exp(g(z) + h(1/z))$ has a wandering domain in $I_e(f)$ which lies in a sector W_α with $0 < \alpha < \pi/2$.



Sketch of the Baker domain construction

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$$f(z)\sim\lambda z$$
 as ${\sf Re}\,z
ightarrow+\infty,$

for some $\lambda > 1$, and $f(\mathbb{H}_r) \subseteq \mathbb{H}_r$ for sufficiently large r. We need

 $|g(z) - \log(\lambda z)| < \varepsilon(|z|)$ on $\mathbb{H}_r = \{z \in \mathbb{C} : \operatorname{Re} z > r\}.$

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The largest error function that we can have is $\varepsilon(t) = \sqrt{2}e^{-t}$.

Sketch of the Baker domain construction II

Arakelian's theorem tells us that we can only construct g with error $\varepsilon(t)$ in a sector of angle $0 < \alpha < \pi$. Thus, we semiconjugate λz with \sqrt{z} :



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The new error function is

$$\widetilde{arepsilon}(t)=rac{arepsilon(2t)}{2}=2\sqrt{2}e^{-2t}.$$

Sketch of the Baker domain construction II

Arakelian's theorem tells us that we can only construct g with error $\varepsilon(t)$ in a sector of angle $0 < \alpha < \pi$. Thus, we semiconjugate λz with \sqrt{z} :



The new error function is

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Therefore we can construct an entire function g such that

$$|g(z) - \log(\sqrt{\lambda z^2})| < \varepsilon(|z|)$$
 on $\sqrt{\mathbb{H}_r} = \{z \in \mathbb{C} : \operatorname{Re} z^2 > r\}.$

Then $f(z) = \exp(g(z))$ satisfies that $f(\sqrt{\mathbb{H}_r}) \subseteq \sqrt{\mathbb{H}_r}$ for sufficiently large r, and hence f has a fixed hyperbolic Baker domain containing $\sqrt{\mathbb{H}_r}$ (that escapes to ∞).

Sketch of the Baker domain construction III

In order to construct a transcendental self-map of \mathbb{C}^* , let $\phi(z) = e^{-2z}$, and use Arakeljan's theorem on tangential approximation to construct an entire function g such that

$$\begin{cases} \left|g(z) - \frac{\log(\lambda z) - n \log z}{\phi(1/z)}\right| < \sqrt{2}e^{-4|z|}, & \text{ on } \sqrt{\mathbb{H}_r}, \\ |g(z)| < 1, & \text{ on } \mathbb{D}. \end{cases}$$

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Then the function

$$f(z) = z^n \exp\left(g(z)\phi(1/z) + g(1/z)\phi(z)\right)$$

is a transcendental self-map of \mathbb{C}^* with two fixed hyperbolic Baker domains, one escaping to 0 and the other one to ∞ .

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Then the function

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is a transcendental self-map of \mathbb{C}^* with two fixed hyperbolic Baker domains, one escaping to 0 and the other one to $\infty.$

Finally, if you want to have a 2-periodic Baker domain U with $e = \overline{\infty 0}$, define

$$\tilde{f}(z) = 1/f(z)$$

so that $U \rightleftharpoons 1/U$ by \tilde{f} .

THE END Thank you for your attention!

